KIRILLOV–RESHETIKHIN CRYSTALS FOR NONEXCEPTIONAL TYPES

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Abstract. We provide combinatorial models for all Kirillov–Reshetikhin crystals of nonexceptional type, which were recently shown to exist. For types $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$ we rely on a previous construction using the Dynkin diagram automorphism which interchanges nodes 0 and 1. For type $C_n^{(1)}$ we use a Dynkin diagram folding and for types $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$ a similarity construction. We also show that for types $C_n^{(1)}$ and $D_{n+1}^{(2)}$ the analog of the Dynkin diagram automorphism exists on the level of crystals.

1. Introduction

Let $\mathfrak{g}$ be an affine Lie algebra and $U_q'(\mathfrak{g})$ the corresponding quantum algebra without derivation. Irreducible finite-dimensional $U_q'(\mathfrak{g})$-modules were classified by Chari and Pressley [3, 4] in terms of Drinfeld polynomials. It was then conjectured by Hatayama et al. [8, 9] that a certain subset of such modules known as Kirillov–Reshetikhin (KR) modules $W_s^{(r)}$ have a crystal basis $B^{r,s}$ in the sense of Kashiwara [19]. Here the index r corresponds to a node of the Dynkin diagram of $\mathfrak{g}$ except the affine node 0 as specified in [16], and s is an arbitrary positive integer. This conjecture was recently confirmed in [28] for all $\mathfrak{g}$ of nonexceptional affine type by using the results [10, 11, 26] on T-systems. (For many special cases including exceptional ones the conjecture was already known to be true in [1, 12, 15, 18, 21, 22, 24, 27, 35].) By the theory of affine finite crystals developed in [17, 18], that is, crystal bases of finite-dimensional $U_q'(\mathfrak{g})$-modules, any integrable highest weight $U_q(\mathfrak{g})$-module can be realized as a semi-infinite tensor product of perfect crystals. This is known as the path realization. Many of the crystals coming from KR modules, KR crystals for short, are (conjectured to be) perfect [8, 9, 18, 31]. By [6, 25] perfect KR crystals are isomorphic as classical crystals to certain Demazure subcrystals of integrable highest weight crystals. In [7] it was shown that under certain assumptions the classical isomorphism from the Demazure crystal to the KR crystal, sends zero arrows to zero arrows. This implies in particular that the affine crystal structure on these KR crystals is unique.

In this paper we solve the long outstanding problem of the construction of KR crystals. We provide an explicit combinatorial crystal structure for all KR crystals $B^{r,s}$ of $W_s^{(r)}$ for $\mathfrak{g}$ of nonexceptional type. To do this we first construct a combinatorial model $V^{r,s}(= V_{\mathfrak{g}}^{r,s})$ for the KR crystal $B^{r,s}$. Let us look at type $A_{n-1}^{(1)}$ for instance. Since it is known that $W_s^{(r)}$ is irreducible as a $U_q(A_{n-1})$-module with highest weight $s\Lambda_r$, the combinatorial crystal $V^{r,s}$ is defined to be the highest weight A_{n-1}-crystal $B(s\Lambda_r)$, which can be identified with the set of semi-standard

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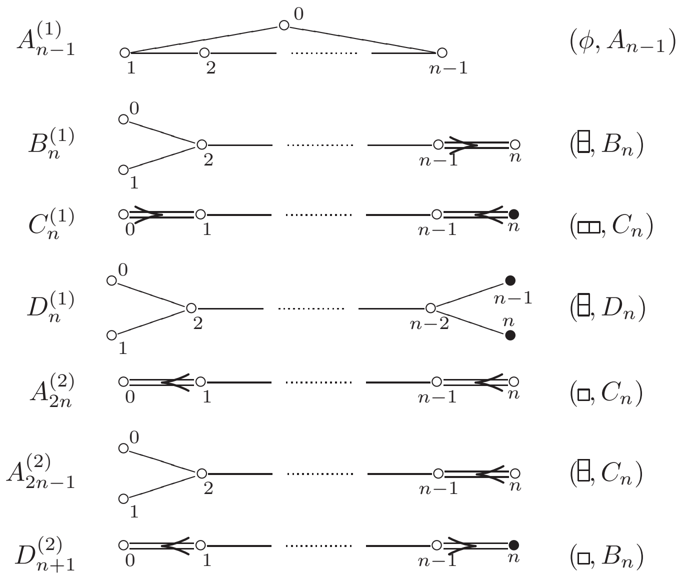

TABLE 1. Dynkin diagrams

tableaux of rectangular shape (s^r). On $B(s\Lambda_r)$ the action of crystal operators e_i, f_i $(i = 1, 2, \ldots, n-1)$ is known [23]. Hence, we are left to define the action of e_0, f_0. This was done by Shimozono [33], exploiting the fact that there is an automorphism σ defined on $B(s\Lambda_r)$ which corresponds to the Dynkin diagram automorphism mapping i to $i+1$ modulo n (see Section 4.1). With this σ the affine crystal operator is given by $e_0 = \sigma^{-1} \circ e_1 \circ \sigma$ and $f_0 = \sigma^{-1} \circ f_1 \circ \sigma$.

For other types of nonexceptional algebras, KR modules are not necessarily irreducible. Let $\mathfrak{g}_0$ be the finite-dimensional simple Lie algebra obtained by removing 0 from the Dynkin diagram of $\mathfrak{g}$. In general $W_s^{(r)}$ decomposes into

$$W_s^{(r)} \simeq \bigoplus_{\lambda} V(\lambda) \tag{1.1}$$

as a $U_q(\mathfrak{g}_0)$-module, where $V(\lambda)$ stands for the irreducible $U_q(\mathfrak{g}_0)$-module with highest weight corresponding to λ and the sum runs over all partitions λ that can be obtained from the $r \times s$ (or $r \times (s/2)$ only when $\mathfrak{g} = B_n^{(1)}$ and $r = n$) rectangle by removing pieces of shape ν (where $(\nu, \mathfrak{g}_0)$ are given in Table 1). There are some exceptions, in which case $W_s^{(r)}$ is irreducible and does not decompose as in (1.1). We call these nodes r "exceptional"; they are the filled nodes in Table 1. [1] These decompositions were proven by Chari [2] in the untwisted cases. In general they can be proven from the results by Nakajima and Hernandez [10, 11, 26]. See [8, 9].

The combinatorial crystal $V^{r,s}$ is constructed according to whether r is exceptional or not. We first treat the nonexceptional cases. Since we already know the $\mathfrak{g}_0$-crystal structure via Kashiwara-Nakashima tableaux [23], it is sufficient to define an appropriate automorphism related to a Dynkin diagram automorphism or an explicit 0-action. For types $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$ we rely on the construction

[1]In [28] the node $r = n$ for type $B_n^{(1)}$ was accidentally marked as exceptional.

in [31] of an automorphism σ which fixes the $\{2,3,\ldots,n\}$-crystal structure and interchanges nodes 0 and 1, and define the affine crystal operator as $e_0 = \sigma \circ e_1 \circ \sigma$ (see Section 4.2). The $C_n^{(1)}$ crystal $V^{r,s}$ is realized as a "virtual" crystal inside the type $A_{2n+1}^{(2)}$ KR crystal $V^{r,s}$ using a folding of the Dynkin diagram (see Section 4.3). For $\mathfrak{g}$ of type $D_{n+1}^{(2)}$ or $A_{2n}^{(2)}$, Kashiwara's similarity method [20] is used to construct $V_{\mathfrak{g}}^{r,s}$ through a unique injective embedding $S : V_{\mathfrak{g}}^{r,s} \to V_{C_n^{(1)}}^{r,s}$ (see Section 4.4). The combinatorial crystals for the exceptional nodes such as $V^{n,s}$ for types $C_n^{(1)}$, $D_{n+1}^{(2)}$ and $V^{n,s}$, $V^{n-1,s}$ for type $D_n^{(1)}$ are treated in Section 6.

The main theorem of this paper can be stated as follows:

Theorem 1.1. *The combinatorial crystal $V^{r,s}$ given in this paper is isomorphic as a $\mathfrak{g}$-crystal to the KR crystal $B^{r,s}$.*

This theorem summarizes Theorem 4.1 for type $A_{n-1}^{(1)}$ shown in [18, 33], Theorem 5.1 for types $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$ shown in [28], Theorem 5.3 for $r = n$ for type $B_n^{(1)}$, Theorem 5.7 for types $C_n^{(1)}$, $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, Theorems 6.3 for the exceptional node $r = n$ of types $C_n^{(1)}$, $D_{n+1}^{(2)}$, and finally Theorem 6.4 for exceptional nodes $r = n-1, n$ of type $D_n^{(1)}$.

The general strategy to deduce $V^{r,s} \cong B^{r,s}$ is to show a certain uniqueness theorem. For example, for types $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$ it is stated as follows. If $V^{r,s}$ and B have the same decompositions as $\{1,2,\ldots,n\}$ and $\{0,2,\ldots,n\}$-crystals, then they have to be isomorphic (see Section 5.1).

By construction, the Dynkin diagram automorphism σ for type $A_{n-1}^{(1)}$, $B_n^{(1)}$, $D_n^{(1)}$, and $A_{2n-1}^{(2)}$ acts on the combinatorial crystal $V^{r,s}$. The Dynkin diagrams for type $C_n^{(1)}$ and $D_{n+1}^{(2)}$ also have an automorphism mapping $i \mapsto n-i$ for all $i \in \{0,1,\ldots,n\}$. However, from the construction of $V^{r,s}$ for these types using Dynkin diagram foldings and similarity methods, it is not obvious that this Dynkin diagram automorphism extends to $V^{r,s}$. This is proven in Theorem 7.1 and shows in particular that [29, Assumption 1] holds.

Organization. The paper is organized as follows. In Section 2 we review some general facts and definitions about crystals, in particular the classical crystals of type B_n, C_n, D_n using Kashiwara–Nakashima tableaux [23]. In Section 3 we review the branching $X_n \to X_{n-1}$ in terms of $\pm$-diagrams, and derive some properties of B_n, C_n crystals and their corresponding $\pm$-diagrams. These definitions and properties are used to define the combinatorial KR crystals $V^{r,s}$ in Section 4 and to show in Section 5 that there is a unique crystal with the classical decompositions of $B^{r,s}$, thereby proving that $V^{r,s} \cong B^{r,s}$. The $V^{r,s}$ for exceptional nodes are treated in Section 6. In Section 7 it is shown that the Dynkin diagram automorphism of type $C_n^{(1)}$ and $D_{n+1}^{(2)}$ extends to $V^{r,s}$.

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2. Some review of crystal theory

We review some basic definitions and facts about crystals that are used in this paper in Section 2.1. In order to describe the crystal graphs for the finite-dimensional modules of quantum groups of classical type, Kashiwara and Nakashima [23] introduced the analogue of semi-standard tableaux, called Kashiwara–Nakashima (KN) tableaux. In Sections 2.2-2.4 we review KN tableaux for types B_n, C_n, and D_n, respectively.

2.1. General definitions.

Crystal theory was introduced by Kashiwara [19] which provides a combinatorial way to study the representation theory of quantum algebras $U_q(\mathfrak{g})$. In this paper $\mathfrak{g}$ stands for a simple Lie algebra or affine Kac–Moody Lie algebra with index set I and $U_q(\mathfrak{g})$ is the corresponding quantum algebra. Axiomatically, a $\mathfrak{g}$-crystal is a nonempty set B together with maps

$$\begin{aligned} e_i, f_i &: B \to B \cup \{\emptyset\} \qquad \text{for } i \in I,\\ \mathrm{wt} &: B \to P, \end{aligned}$$

where P is the weight lattice associated to $\mathfrak{g}$. The maps e_i and f_i are Kashiwara's crystal operators and wt is the weight function. Stembridge [34] gave a local characterization to determine when an axiomatic crystal actually corresponds to a $U_q(\mathfrak{g})$-representation when $\mathfrak{g}$ is simply-laced. For further details about crystal theory, please consult for example [19, 13].

To each crystal one can associate a crystal graph with vertices in B and an arrow colored $i \in I$ from b to b' if $f_i(b) = b'$. For $b \in B$ and $i \in I$, let

$$\begin{aligned} \varepsilon_i(b) &= \max\{k \in \mathbb{Z}_{\geq 0} \mid e_i^k(b) \neq \emptyset\},\\ \varphi_i(b) &= \max\{k \in \mathbb{Z}_{\geq 0} \mid f_i^k(b) \neq \emptyset\}. \end{aligned}$$

An element $b \in B$ is called highest (resp. lowest) weight if $e_i(b) = \emptyset$ (resp. $f_i(b) = \emptyset$) for all $i \in I$. For $J \subset I$, we say that $b \in B$ is J-highest (resp. J-lowest) if $e_i(b) = \emptyset$ (resp. $f_i(b) = \emptyset$) for all $i \in J$.

We say that $b, b' \in B$ are J-related or $b \sim_J b'$ in symbols, if there exist J-highest elements b_0, b'_0 of the same weight such that $b = f_{\vec{c}}(b_0)$, $b' = f_{\vec{c}}(b'_0)$ for some sequence $\vec{c}$ from J. Here $f_{\vec{c}} = f_{c_1} \cdots f_{c_\ell}$ for $\vec{c} = (c_1, \ldots, c_\ell)$. A J-component $\mathcal{C}$ of a crystal B is a connected component in the crystal graph of B when only considering arrows colored $i \in J$.

We denote by $B(\Lambda)$ the highest weight crystal of highest weight Λ, where Λ is a dominant integral weight. Let Λ_i with $i \in I$ be the fundamental weights associated to a simple Lie algebra of classical types, that is, A_{n-1}, B_n, C_n or D_n. Then as usual, a dominant integral weight $\Lambda = \Lambda_{i_1} + \cdots + \Lambda_{i_k}$ is identified with a partition with columns of height i_j for $1 \leq j \leq k$, except when Λ_{i_j} is a spin weight, in which case we identify Λ_{i_j} with a column of height n and width $1/2$. For type B_n the

fundamental weight Λ_n is a spin weight and for type D_n the fundamental weights Λ_{n-1} and Λ_n are spin weights. In this paper we use French notation where parts are drawn in increasing order from top to bottom. For type A_{n-1}, the highest weight crystal $B(\Lambda)$ is given by the set of all semi-standard Young tableaux of shape Λ over the alphabet $\{1, 2, \dots, n\}$. For types B_n, C_n, and D_n elements in $B(\Lambda)$ are given by Kashiwara–Nakashima (KN) tableaux [23]; they are reviewed in the next subsections.

Let B_1, B_2 be crystals. Then $B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$ can be endowed with the structure of crystal. In order to compute the action of e_i, f_i on multiple tensor products, it is convenient to use the rule called "signature rule". Let $b_1 \otimes b_2 \otimes \cdots \otimes b_m$ be an element of the tensor product of crystals $B_1 \otimes B_2 \otimes \cdots \otimes B_m$. One wishes to find the indices j, j' such that

$$\begin{aligned} e_i(b_1 \otimes \cdots \otimes b_m) &= b_1 \otimes \cdots \otimes e_i b_j \otimes \cdots \otimes b_m, \\ f_i(b_1 \otimes \cdots \otimes b_m) &= b_1 \otimes \cdots \otimes f_i b_{j'} \otimes \cdots \otimes b_m. \end{aligned}$$

To do it, we introduce (i-)signature by

$$\overbrace{-\cdots-}^{\varepsilon_i(b_1)}\overbrace{+\cdots+}^{\varphi_i(b_1)}\overbrace{-\cdots-}^{\varepsilon_i(b_2)}\overbrace{+\cdots+}^{\varphi_i(b_2)}\ \cdots\cdots\ \overbrace{-\cdots-}^{\varepsilon_i(b_m)}\overbrace{+\cdots+}^{\varphi_i(b_m)}.$$

We then reduce the signature by deleting the adjacent $+-$ pair successively. Eventually we obtain a reduced signature of the following form.

$$--\cdots-++\cdots+$$

Then the action of e_i (resp. f_i) corresponds to changing the rightmost $-$ to $+$ (resp. leftmost $+$ to $-$). If there is no $-$ (resp. $+$) in the signature, then the action of e_i (resp. f_i) should be set to $\emptyset$. The value of $\varepsilon_i(b)$ (resp. $\varphi_i(b)$) is given by the number of $-$ (resp. $+$) in the reduced signature.

Consider, for instance, an element $b_1 \otimes b_2 \otimes b_3$ of the 3 fold tensor product $B_1 \otimes B_2 \otimes B_3$. Suppose $\varepsilon_i(b_1) = 1, \varphi_i(b_1) = 2, \varepsilon_i(b_2) = 1, \varphi_i(b_2) = 1, \varepsilon_i(b_3) = 2, \varphi_i(b_3) = 1$. Then the signature and reduced one read

$$\begin{array}{cccccc} \text{sig} & -++ & \cdot & -+ & \cdot & --+ \\ \text{red sig} & - & \cdot & & \cdot & +. \end{array}$$

Thus we have

$$\begin{aligned} e_i(b_1 \otimes b_2 \otimes b_3) &= e_i b_1 \otimes b_2 \otimes b_3, \\ f_i(b_1 \otimes b_2 \otimes b_3) &= b_1 \otimes b_2 \otimes f_i b_3. \end{aligned}$$

2.2. **KN tableaux of type C_n.** In this section we review KN tableaux of type C_n. On the set of letters $\{i, \bar{i} \mid 1 \le i \le n\}$, introduce the following order

$$1 \prec 2 \prec \cdots \prec n \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1}.$$

As a set the crystal $B(\Lambda_N)$ of the fundamental representation with highest weight Λ_N is given by

$$B(\Lambda_N) = \left\{ \begin{array}{|c|} \hline i_N \\ \hline \vdots \\ \hline i_1 \\ \hline \end{array} \;\middle|\; \begin{array}{l} (1)\ 1 \preceq i_1 \prec \cdots \prec i_N \preceq \overline{1}, \\ (2)\ \text{if } i_k = p \text{ and } i_l = \overline{p}, \text{ then } k + (N - l + 1) \le p \end{array} \right\}. \tag{2.1}$$

To describe the crystal $B(\Lambda_M + \Lambda_N)$ $(M \ge N)$ we need to define the notion of (a, b)-configurations.

Definition 2.1. Let

$$u = \begin{array}{|c|}\hline i_M \\ \hline \vdots \\ \hline i_1 \\ \hline \end{array} \in B(\Lambda_M) \quad \text{and} \quad v = \begin{array}{|c|}\hline j_N \\ \hline \vdots \\ \hline j_1 \\ \hline \end{array} \in B(\Lambda_N).$$

For $1 \leq a \leq b \leq n$, we say $w = (u, v)$ is in the (a, b)-configuration if it satisfies the following: There exist $1 \leq p \leq q < r \leq s \leq N$ such that $i_p = a, i_q = b, i_r = \overline{b}, j_s = \overline{a}$ or $i_p = a, j_q = b, j_r = \overline{b}, j_s = \overline{a}$. The definition includes the case where $a = b$, $p = q$, and $r = s$. Define

$$p(a, b; w) = (q - p) + (s - r).$$

Then the crystal $B(\Lambda_M + \Lambda_N)$ of the highest weight module of highest weight $\Lambda_M + \Lambda_N$ is given by

(2.2)

$$B(\Lambda_M + \Lambda_N) = \left\{ w = \begin{array}{|c|c|}\hline i_M & \multicolumn{1}{c}{} \\ \hline \vdots & j_N \\ \hline \vdots & \vdots \\ \hline i_1 & j_1 \\ \hline \end{array} \;\middle|\; \begin{array}{l} (1)\ i_k \preceq j_k \text{ for } 1 \leq k \leq N, \\ (2) \text{ if } w \text{ is in the } (a,b)\text{-configuration,} \\ \qquad \text{then } p(a, b; w) < b - a \end{array} \right\}.$$

Note that an element of $B(\Lambda_M + \Lambda_N)$ cannot be in the (a, a)-configuration. We can now describe the crystal $B(\Lambda)$ of the highest weight module of highest weight $\Lambda = \Lambda_{l_1} + \cdots + \Lambda_{l_p}$ $(n \geq l_1 \geq \cdots \geq l_p \geq 1)$ as

(2.3)

$$B(\Lambda) = \left\{ w = \begin{array}{|c|c|c|c|}\hline t_1 & \vdots & \vdots & t_p \\ \hline \end{array} \;\middle|\; \begin{array}{|c|c|}\hline t_k & t_{k+1} \\ \hline \end{array} \in B(\Lambda_{l_k} + \Lambda_{l_{k+1}}) \text{ for any } k = 1, \dots, p-1 \right\}.$$

Let us describe the action of crystal operators on $B(\Lambda)$. For the simplest case $B(\Lambda_1)$ they are given by the following crystal graph.

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\overline{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\overline{2}} \xrightarrow{1} \boxed{\overline{1}}$$

For the general case, we regard a tableau as an element of $B(\Lambda_1)^{\otimes N}$, where N is the number of boxes of the tableau. We move along the tableau from the rightmost column to left, and in each column we move from bottom to top. Then we obtain the sequence of letters $b_1, b_2, \dots, b_N$. We associate the tableau to $b_1 \otimes b_2 \otimes \cdots \otimes b_N$ in $B(\Lambda_1)^{\otimes N}$. Then the action of e_i, f_i is given by the multiple tensor product rule explained in the previous subsection.

2.3. **KN tableaux of type B_n.** The construction for B_n is divided into two cases. Set $\omega_i = \Lambda_i$ (for $i \leq n-1$), $\omega_n = 2\Lambda_n$, and on the set of letters $\{i, \overline{i} \mid 1 \leq i \leq n\} \cup \{0\}$ introduce the following order

$$1 \prec 2 \prec \cdots \prec n \prec 0 \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1}.$$

As a set the crystal $B(\omega_N)$ of the fundamental representation with highest weight ω_N is given by

(2.4)
$$B(\omega_N) = \left\{ \begin{array}{|c|} \hline i_N \\ \hline \vdots \\ \hline i_1 \\ \hline \end{array} \;\middle|\; \begin{array}{l} (1)\ 1 \preceq i_1 \prec \cdots \prec i_N \preceq \overline{1}, \\ \text{ but no element other than 0 can appear more than once} \\ (2)\ \text{if } i_k = p \text{ and } i_l = \overline{p}, \text{ then } k + (N - l + 1) \leq p \end{array} \right\}.$$

The second case is the "spin representation" of highest weight Λ_n. Define on $\{1, \ldots, n, \overline{n}, \ldots, \overline{1}\}$ a linear order by

$$1 \prec 2 \prec \cdots \prec n \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1}.$$

Then as a set the crystal $B(\Lambda_n)$ is given by

$$B(\Lambda_n) = \left\{ \begin{array}{|c|} \hline i_n \\ \hline \vdots \\ \hline i_1 \\ \hline \end{array} \;\middle|\; \begin{array}{l} (1)\ 1 \preceq i_1 \prec \cdots \prec i_n \preceq \overline{1}, \\ (2)\ i \text{ and } \overline{i} \text{ do not appear simultaneously} \end{array} \right\}. \tag{2.5}$$

To describe the crystal $B(\omega_M + \omega_N)$ $(M \geq N)$ we again need to define the notion of (a, b)-configurations.

Definition 2.2. Let

$$u = \begin{array}{|c|} \hline i_M \\ \hline \vdots \\ \hline i_1 \\ \hline \end{array} \in B(\omega_M) \quad \text{and} \quad v = \begin{array}{|c|} \hline j_N \\ \hline \vdots \\ \hline j_1 \\ \hline \end{array} \in B(\omega_N).$$

For $1 \leq a \leq b < n$ we have the same conditions as in Definition 2.1 for type C_n. For $1 \leq a < n$, we say $w = (u, v)$ is in the (a, n)-configuration if it satisfies the following: There exist $1 \leq p \leq q < r = q + 1 \leq s \leq N$ such that $i_p = a$, $j_s = \overline{a}$ and one of the conditions is satisfied:

(1) i_q and $i_r (= i_{q+1})$ are $n, 0$, or $\overline{n}$.
(2) j_q and $j_r (= j_{q+1})$ are $n, 0$, or $\overline{n}$.

We say $w = (u, v)$ is in the (n, n)-configuration if there are $1 \leq p < q \leq N$ such that $i_p = n$ or 0 and $j_p = 0$ or $\overline{n}$. Define again

$$p(a, b; w) = (q - p) + (s - r).$$

Then as a set the crystal $B(\omega_M + \omega_N)$ of the highest weight module of highest weight $\omega_M + \omega_N$ is given by

(2.6)
$$B(\omega_M + \omega_N) = \left\{ w = \begin{array}{|c|c} \hline i_M & \\ \hline \vdots & \multicolumn{1}{|c|}{j_N} \\ \hline \vdots & \multicolumn{1}{|c|}{\vdots} \\ \hline i_1 & \multicolumn{1}{|c|}{j_1} \\ \hline \end{array} \;\middle|\; \begin{array}{l} (1)\ i_k \preceq j_k \text{ for } 1 \leq k \leq N, \\ \quad \text{and } i_k \text{ and } j_k \text{ cannot both be } 0, \\ (2)\ \text{if } w \text{ is in the } (a, b)\text{-configuration,} \\ \quad \text{then } p(a, b; w) < b - a \end{array} \right\}.$$

Note that an element of $B(\omega_M + \omega_N)$ cannot be in the (a, a)-configuration. The conditions for $B(\Lambda_n + \omega_N)$ are formally just the same as in (2.6), although there is no 0 in the first column.

We can now describe the crystal $B(\Lambda)$ of the highest weight module of highest weight Λ. If Λ is of the form $\Lambda = \omega_{l_1} + \cdots + \omega_{l_p}$ $(n \geq l_1 \geq \cdots \geq l_p \geq 1)$,

(2.7)
$$B(\Lambda) = \left\{ w = \begin{array}{|c|c|c|c|} \hline t_1 & \vdots & \vdots & t_p \\ \hline \end{array} \;\middle|\; \begin{array}{|c|c|} \hline t_k & t_{k+1} \\ \hline \end{array} \in B(\omega_{l_k} + \omega_{l_{k+1}}) \text{ for } k \leq p-1 \right\}.$$

The crystal structure can be described in the same way as in Section 2.2. The only difference is that we have to replace the crystal graph of $B(\Lambda_1)$ with the one below.

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\overline{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\overline{2}} \xrightarrow{1} \boxed{\overline{1}}$$

Else, Λ can be written as $\Lambda = \Lambda_n + \omega_{l_2} + \cdots + \omega_{l_p}$ $(n \geq l_2 \geq \cdots \geq l_p \geq 1)$. In this case,

(2.8)
$$B(\Lambda) = \left\{ w = \begin{array}{|c|c|c|c|} \hline t_1 & \vdots & \vdots & t_p \\ \hline \end{array} \;\middle|\; \begin{array}{|c|c|} \hline t_k & t_{k+1} \\ \hline \end{array} \in \begin{array}{l} B(\Lambda_n + \omega_{l_2}) \text{ for } k = 1 \\ B(\omega_{l_k} + \omega_{l_{k+1}}) \text{ for } 2 \leq k \leq p-1 \end{array} \right\}.$$

The crystal structure in this case is given as that on the tensor product $B(\omega_{l_2} + \cdots + \omega_{l_p}) \otimes B(\Lambda_n)$. For the crystal structure on $B(\Lambda_n)$ see [23, Section 5.4].

2.4. **KN tableaux of type D_n.** For type D_n we only consider $B(\Lambda)$, where the coefficients of Λ_{n-1} and Λ_n of Λ are 0. On the set $\{i, \overline{i} \mid 1 \leq i \leq n\}$ we define the following order

$$1 \prec 2 \prec \ldots \prec n-1 \prec \begin{array}{c} n \\ \overline{n} \end{array} \prec \overline{n-1} \prec \ldots \prec \overline{1}$$

where there is no order between n and $\overline{n}$. Then as a set the crystal of highest weight Λ_N for $N \leq n-2$ is

(2.9)
$$B(\Lambda_N) = \left\{ \begin{array}{|c|} \hline i_N \\ \hline \vdots \\ \hline i_1 \\ \hline \end{array} \;\middle|\; \begin{array}{l} (1)\ i_j \not\succeq i_{j+1} \text{ for } 1 \leq j < N \\ (2) \text{ if } i_k = p \text{ and } i_l = \overline{p}\ (1 \leq p \leq n), \text{ then } k + (N - l + 1) \leq p \end{array} \right\}.$$

To describe the crystal $B(\Lambda_M + \Lambda_N)$ $(M \geq N)$ we need to define again the notion of (a, b)-configurations.

Definition 2.3. Let

$$u = \begin{array}{|c|} \hline i_M \\ \hline \vdots \\ \hline i_1 \\ \hline \end{array} \in B(\Lambda_M) \quad \text{and} \quad v = \begin{array}{|c|} \hline j_N \\ \hline \vdots \\ \hline j_1 \\ \hline \end{array} \in B(\Lambda_N).$$

(1) For $1 \leq a \leq b < n$ we have the same conditions as in Definition 2.1.
(2) For $1 \leq a < n$, we say $w = (u, v)$ is in the (a, n)-configuration if it satisfies the following: There exist $1 \leq p \leq q < r = q + 1 \leq s \leq N$ such that $i_p = a, j_s = \overline{a}$ and one of the conditions is satisfied:
   (a) i_q and $i_r (= i_{q+1})$ are n or $\overline{n}$,

   (b) j_q and $j_r(=j_{q+1})$ are n or $\overline{n}$.
(3) We say $w=(u,v)$ is in the (n,n)-configuration if there are $1 \le p < q \le N$ such that $i_p = n$ or $\overline{n}$ and $j_p = n$ or $\overline{n}$.
(4) For $1 \le a < n$, $w=(u,v)$ is in the a-odd-configuration if the following conditions are satisfied: There exists $1 \le p \le q < r \le s \le N$ such that
   (a) $r-q+1$ is odd,
   (b) $i_p = a$ and $j_s = \overline{a}$,
   (c) $j_q = n, i_r = \overline{n}$ or $j_q = \overline{n}, i_r = n$.
(5) For $1 \le a < n$, $w=(u,v)$ is in the a-even-configuration if the following conditions are satisfied: There exists $1 \le p \le q < r \le s \le N$ such that
   (a) $r-q+1$ is even,
   (b) $i_p = a$ and $j_s = \overline{a}$,
   (c) $j_q = n, i_r = n$ or $j_q = \overline{n}, i_r = \overline{n}$.

Then

(1) If w is in the (a,b)-configuration for $1 \le a \le b \le n$, we define $p(a,b;w) = (q-p)+(s-r)$. If $a=b=n$, set $p(a,b;w)=0$.
(2) If w is in the a-odd or a-even-configuration, we define $q(a;w) = s-p$.

Then the crystal $B(\Lambda_M + \Lambda_N)$ of the highest weight module of highest weight $\Lambda_M + \Lambda_N$ $(n \ge M \ge N \ge 1)$ is given by
(2.10)

$$B(\Lambda_M+\Lambda_N) = \left\{ w = \begin{array}{|c|c|} \hline i_M & \\ \hline \vdots & j_N \\ \hline \vdots & \vdots \\ \hline i_1 & j_1 \\ \hline \end{array} \;\middle|\; \begin{array}{l} (1)\ i_k \preceq j_k \text{ for } 1 \le k \le N, \\ (2)\ \text{if } w \text{ is in the } (a,b)\text{-configuration,} \\ \quad\ \text{then } p(a,b;w) < b-a \\ (3)\ \text{if } w \text{ is in the } a\text{-odd-} \\ \quad\ \text{or } a\text{-even-configuration,} \\ \quad\ \text{then } q(a;w) < n-a \end{array} \right\}.$$

The crystal $B(\Lambda)$ with $\Lambda = \Lambda_{l_1} + \cdots + \Lambda_{l_p}$ $(n-2 \ge l_1 \ge l_2 \ge \cdots \ge l_p \ge 1)$ is described again as (2.3). The crystal structure on $B(\Lambda)$ is obtained as in Section 2.2. The only difference is that we have to replace the crystal graph of $B(\Lambda_1)$ with the one below.

$$\begin{tikzcd}
 & & & \boxed{n} \arrow[dr, "n"] & & & \\
\boxed{1} \arrow[r, "1"] & \cdots \arrow[r, "n-2"] & \boxed{n-1} \arrow[ur, "n-1"] \arrow[dr, "n"'] & & \boxed{\overline{n-1}} \arrow[r, "n-1"] & \cdots \arrow[r, "1"] & \boxed{\overline{1}} \\
 & & & \boxed{\overline{n}} \arrow[ur, "n-1"'] & & &
\end{tikzcd}$$

3. Properties and branching of classical crystals

In this section we derive some properties of B_n, C_n crystals in Section 3.1, branching rules $X_n \to X_{n-1}$ in terms of $\pm$-diagrams where $X = B, C, D$ in Section 3.2, and properties of $\pm$-diagrams for type B_n, C_n in Section 3.3. These results will be used later in the construction of KR crystals.

3.1. **Properties of B_n and C_n crystals.** In this section we prove some preliminary results for the form and properties of special elements in $B(\Lambda)$ of type C_n and B_n. In addition to the index set of the classical Dynkin diagram $I_0 = \{1,2,\ldots,n\}$, we will also use $J = \{1,2,\ldots,n-1\}$ and $J' = \{2,3,\ldots,n\}$.

Lemma 3.1. *Let b be a J-lowest weight element of type C_n or B_n. Then the columns of b must be of the form*

$$\begin{array}{|c|}\hline \bar{l}_t \\ \hline \vdots \\ \hline \bar{l}_1 \\ \hline 0^\alpha \\ \hline n \\ \hline n-1 \\ \hline \vdots \\ \hline k \\ \hline \end{array} \tag{3.1}$$

where $l_i < k$ for all $1 \le i \le t$, $\alpha \ge 0$, and $\alpha = 0$ for type C_n.

Proof. We prove this lemma for C_n. The B_n case can be treated in exactly the same way.

We prove the lemma by induction on the columns (from left to right).

All unbarred letters of b form a tableau in the bottom left (in French convention). If there is no unbarred letter in the first column, then there is none in the entire tableau. Suppose the letter k occurs in the first column, then there has to be (since b is lowest weight) a $k+1$ or a $\overline{k}$ in the same column. Suppose there is a $\overline{k}$. Then, since b is lowest weight, there has to be a $\overline{k-1}$. Then the top of the column has the form

$$\begin{array}{|c|}\hline \bar{1} \\ \hline \bar{2} \\ \hline \vdots \\ \hline \bar{k} \\ \hline \vdots \\ \hline \end{array}.$$

But this is not a legal type C_n tableau, since the pair k and $\overline{k}$ does not satisfy the condition of (2.1). Hence there must be a $k+1$. Therefore the first column looks as follows

$$\begin{array}{|c|}\hline \bar{l}_r \\ \hline \vdots \\ \hline \bar{l}_1 \\ \hline n \\ \hline \vdots \\ \hline k+1 \\ \hline k \\ \hline \end{array}.$$

The condition $l_i < k$ for all i is forced again by the tableau rules (2.1) for C_n, since the top has to be consecutive in the barred letters for b to be J-lowest weight. This concludes the induction beginning. Now suppose the claim is true for the first $m-1$ columns. If the m-th column does not start with an unbarred letter, we are done. So suppose it starts with a k.

Case 1: The column $m-1$ starts with a k as well. Since b is J-lowest weight, there must be either a $k+1$ or a $\overline{k}$ to bracket the k in column m. By induction, in

every column to the left of the m-th column, there is no $\overline{k}$. Suppose there is a $\overline{k}$ in the m-th column. Then b contains the following pattern:

$$\begin{array}{|c|c|} \hline * & \overline{k} \\ \hline \vdots & \vdots \\ \hline k & k \\ \hline \end{array} \tag{3.2}$$

which is illegal by (2.2). Hence there must be a $k+1$. This letter $k+1$ has to be bracketed again, since b is lowest weight. Suppose it is bracketed with $\overline{k+1}$. But by induction to the left there is no unbracketed $k+1$ or $\overline{k}$, which is a contradiction. Hence there is a $k+2$ and so on.

Case 2: There is no column to the left of the m-th column, that starts with a k. Suppose k is bracketed with a $\overline{k}$. This $\overline{k}$ has to be in the same column as k, since by induction, there is no $\overline{k}$ to the left. But this $\overline{k}$ has to be bracketed by a $\overline{k-1}$ which is in the same column (by induction). Hence this column is of the form

$$\begin{array}{|c|} \hline \overline{k-i} \\ \hline \vdots \\ \hline \overline{k-1} \\ \hline \overline{k} \\ \hline \vdots \\ \hline k \\ \hline \end{array}$$

for some i. The letter $\overline{k-i}$ has to be bracketed as well. Suppose it is bracketed with a $k-i$. Then we obtain an illegal C_n tableau by (2.2). If it is bracketed with a $\overline{k-i-1}$ in another column, then there is a corresponding unbarred letter in this column. This letter has to be bigger or equal to $k-i$, which gives again an illegal tableau. The column

$$\begin{array}{|c|} \hline \overline{1} \\ \hline \vdots \\ \hline \overline{k-1} \\ \hline \overline{k} \\ \hline \vdots \\ \hline k \\ \hline \end{array}$$

is also illegal by (2.1). Hence k has to be bracketed by $k+1$. Repeating the arguments as before one obtains that the m-th column is of the form

$$\begin{array}{|c|} \hline \vdots \\ \hline n \\ \hline \vdots \\ \hline k+1 \\ \hline k \\ \hline \end{array}$$

To finish the induction, we need to show that above the n there are only $\overline{l}_i$ with $l_i < k$. Suppose there is an $\overline{l}$ with $l \geq k$. Then, since the element is lowest weight, $\overline{l}$ must be bracketed. It cannot be bracketed with an unbarred letter to the left of

the m-th column, since they are all smaller than k (if there is an unbarred letter $l = k$, then the tableau has a pattern of the form (3.2), which is illegal by (2.2)). By induction the only barred letters to the left of the m-th column are $\overline{p}$, with $p < k - 1$. (If there is a k at the left of the m-th column, then there could not be a $\overline{k-1}$, because the tableau would be again illegal). Hence all barred letters in the m-th column must be smaller than k. □

The inner shape of a J-lowest weight element b is defined to be the shape after deleting all $\overline{1}$s.

Lemma 3.2. *The highest weight of the J'-component of a J-lowest weight element b is given by the inner shape of b.*

Proof. By Lemma 3.1, b contains no 1's. Construct a tableau b' from b by deleting all $\overline{1}$'s and replacing each letter c (resp. $\overline{c}$) by $c-1$ (resp. $\overline{c-1}$). Then one finds that b' is again a KN tableau for C_{n-1} (B_{n-1} resp.): To be a KN tableau there are conditions in each column (2.1) (resp. (2.4)) and those for adjacent columns (2.2) (resp. (2.6)). The former condition is satisfied, since there is no $(k, \overline{k})$ pair by Lemma 3.1. The latter condition is invariant under changing contents as above. So by applying a sequence of e_i with $i \in \{1, 2, \ldots, n-1\}$, b' can be raised to the C_{n-1}-highest weight element (B_{n-1} resp.). □

Recall that $b, b' \in B(\Lambda)$ are J'-related or $b \sim_{J'} b'$ in symbols, if there exist J'-highest elements b_0, b'_0 of the same weight such that $b = f_{\vec{c}}(b_0)$, $b' = f_{\vec{c}}(b'_0)$ for some sequence $\vec{c}$ from J'.

Corollary 3.3. *If b_1 and b_2 are J-lowest, $b_1 \sim_{J'} b_2$ and $\mathrm{wt}\,(b_1) = \mathrm{wt}\,(b_2)$, then b_1 and b_2 differ just by boxes containing $\overline{1}$. The inner tableaux are the same.*

Proof. This follows directly from Lemma 3.2. □

3.2. $X_n \to X_{n-1}$ **branching and $\pm$-diagrams.** Let $\mathfrak{g}_0$ be a finite Lie algebra of type $X_n = D_n, B_n$, or C_n. In this section we describe a branching rule for $X_n \to X_{n-1}$ involving $\pm$-diagrams.

A $\pm$-diagram P of shape Λ/λ is a sequence of partitions $\lambda \subset \mu \subset \Lambda$ such that Λ/μ and μ/λ are horizontal strips (i.e. every column contains at most one box). We depict this $\pm$-diagram by the skew tableau of shape Λ/λ in which the cells of μ/λ are filled with the symbol $+$ and those of Λ/μ are filled with the symbol $-$. Write $\Lambda = \mathrm{outer}(P)$ and $\lambda = \mathrm{inner}(P)$ for the outer and inner shapes of the $\pm$-diagram P. When drawing partitions or tableaux, we use the French convention where the parts are drawn in increasing order from top to bottom.

There are a couple further type-specific requirements:

(1) For type C_n the outer shape Λ contains columns of height at most n, but the inner shape λ is not allowed to be of height n (hence there are no empty columns of height n).
(2) For type B_n the outer shape Λ contains columns of height at most n; for the columns of height n, the $\pm$-diagram can contain at most one 0 between $+$ and $-$ at height n and no empty columns are allowed; furthermore there may be a spin column of height n and width 1/2 containing $+$ or $-$.
(3) For type D_n suppose $\Lambda = k_1\Lambda_1 + \cdots + k_{n-1}\Lambda_{n-1} + k_n\Lambda_n$. If $k_n \geq k_{n-1}$ we depict this weight by $(k_n - k_{n-1})/2$ columns of height n colored 1 (where we interpret a 1/2 column as a Λ_n spin column if $k_n - k_{n-1}$ is odd), k_{n-1}

columns of height $n-1$, and as usual k_i columns of height i for $1 \le i \le n-2$. If $k_n < k_{n-1}$ we depict this weight by $(k_{n-1}-k_n)/2$ columns of height n colored 2 (where we interpret a 1/2 column as a Λ_{n-1} spin column if $k_{n-1}-k_n$ is odd), k_n columns of height $n-1$, and as usual k_i columns of height i for $1 \le i \le n-2$. We require that columns of height n are colored, contain $+$, $-$, or $\mp$, but cannot simultaneously contain $+$ and $-$; spin columns can only contain $+$ or $-$.

Proposition 3.4. *For an X_n dominant weight Λ, there is an isomorphism of X_{n-1}-crystals*

$$B_{X_n}(\Lambda) \cong \bigoplus_{\substack{\pm\text{-diagrams } P \\ \mathrm{outer}(P)=\Lambda}} B_{X_{n-1}}(\mathrm{inner}(P)).$$

That is, the multiplicity of $B_{X_{n-1}}(\lambda)$ in $B_{X_n}(\Lambda)$, is the number of $\pm$-diagrams of shape Λ/λ.

Proof. This follows directly from the branching rules for X_n to X_{n-1} (see for example [5, pg. 426]). $\square$

There is a bijection $\Phi : P \mapsto b$ from $\pm$-diagrams P of shape Λ/λ to the set of X_{n-1}-highest weight vectors b of X_{n-1}-weight λ. Namely, we construct a string of operators $f_{\vec{\mathbf{a}}} := f_{a_1} f_{a_2} \cdots f_{a_\ell}$ such that $\Phi(P) = f_{\vec{\mathbf{a}}} u$, where u is the highest weight vector in $B_{X_n}(\Lambda)$. Start with $\vec{\mathbf{a}} = ()$.

(1) Scan the columns of P from right to left. For each column of P for which a $+$ can be added, append $(1,2,\dots,h)$ to $\vec{\mathbf{a}}$, where h is the height of the added $+$. Note that a $+$ is not addable to a spin column. Further type specific rules are:
   (a) For D_n, if there is an addable $+$ at height $n-1$ to a column of height n, append $(1,2,\dots,n-2,n)$ if the color is 1 and $(1,2,\dots,n-1)$ if the color is 2.
   (b) For B_n, if there is a column of height n containing 0 append $(1,2,\dots,n)$ to $\vec{\mathbf{a}}$.

(2) Next scan P from left to right for columns containing a $-$ at height h.
   (a) For D_n, if $h = n$, append $(1,2,\dots,n-2,n)$ if the color is 1 and $(1,2,\dots,n-1)$ if the color is 2 (this also applies to spin columns). If $h = n-1$, append $(1,2,\dots,n)$. For $h < n-1$, append the string $(1,2,\dots,n,n-2,n-3,\dots,h)$ to $\vec{\mathbf{a}}$.
   (b) For B_n, if the $-$ is in a spin column, append $(1,2,\dots,n)$ to $\vec{\mathbf{a}}$. Otherwise append the string $(1,2,\dots,n-1,n,n,n-1,\dots,h)$ to $\vec{\mathbf{a}}$.
   (c) For C_n, append the string $(1,2,\dots,n-1,n,n-1,\dots,h)$ to $\vec{\mathbf{a}}$.

This correspondence can easily be checked by explicitly writing down the X_{n-1}-highest weight vectors for each type.

Alternatively, suppose Λ is a dominant weight; we require that Λ does not contain any columns of height n for type D_n. Then the bijection $\Phi : P \mapsto b$ from $\pm$-diagrams P of shape Λ/λ to the set of X_{n-1}-highest weight vectors b of X_{n-1}-weight λ is as follows. For any columns of height n containing $+$, place a column $12\dots n$ (this includes spin columns for type B_n). Otherwise, place $\overline{1}$ in all positions in P that contain a $-$, place a 0 in the position containing 0, and fill the remainder of all

columns by strings of the form $23\dots k$. We move through the columns of b from top to bottom, left to right. Each $+$ in P (starting with the leftmost moving to the right ignoring $+$ at height n) will alter b as we move through the columns. Suppose the $+$ is at height h in P. If one encounters a spin column of type B_n, replace it by a column $12\dots h\, h+2\dots n\, \overline{h+1}$ (read from bottom to top). Otherwise, if one encounters a $\overline{1}$, replace $\overline{1}$ by $\overline{h+1}$. If one encounters a 2, replace the string $23\dots k$ by $12\dots h\, h+2\dots k$.

3.3. **Properties of $\pm$-diagrams for type B_n and C_n.** Let $B(\Lambda)$ (resp. $B_{\mathfrak{g}_0}(\Lambda)$) be the C_n (resp. $\mathfrak{g}_0$)-crystal of the highest weight module of highest weight Λ and $I_0 = \{1, 2, \dots, n\}$. Define

$$(m_1, m_2, \dots, m_n) = \begin{cases} (2, \dots, 2, 2) & \text{for type } C_n, \\ (2, \dots, 2, 1) & \text{for type } B_n. \end{cases}$$

By [20, Theorem 5.1] with $\xi = \mathrm{id}$, there exists a unique injective map $\overline{S} : B_{\mathfrak{g}_0}(\Lambda) \to B(2\Lambda)$ such that $\overline{S}(e_i b) = e_i^{m_i}\overline{S}(b)$ and $\overline{S}(f_i b) = f_i^{m_i}\overline{S}(b)$ for $i \in I_0$.

Lemma 3.5. *Let $\Lambda = \sum_{i=1}^n k_i\Lambda_i$, $b \in B_{\mathfrak{g}_0}(\Lambda)$ be a $\{2, \dots, n\}$-highest element, and P the corresponding $\pm$-diagram. Then the $\pm$-diagram corresponding to $\overline{S}(b)$ in $B(\hat{\Lambda})$, where $\hat{\Lambda} = \sum_{i=1}^n m_i k_i \Lambda_i$, is obtained by doubling each column of P together with its signs. This doubling procedure needs special treatment when a sign is in a spin column or 0 is at height n for type B_n. If the former occurs, we replace the spin column with a full column of the same sign. If the latter occurs, we replace it with two columns containing $+$ and $-$.*

Proof. First we prove the claim when P contains only $-$ except at height n. At height n we allow the sign to be $+$ or 0. Let λ_i be the length of the i-th row of the inner shape minus the number of columns of height n with $+$, and μ_i the number of $-$'s in the i-th row. If there is a spin column in type B_n, we regard its length or number of $-$ (if it exists) to be 1/2. Let ν be the number of columns with 0. $\nu = 0$ for type C_n and 0 or 1 for type B_n. Set $M = \mu_1 + \mu_2 + \dots + \mu_n$. Let u be the $\{1, 2, \dots, n\}$-highest element vector of weight given by the outer shape of P. Set

$$\vec{a} = (1^{M+\lambda_1} 2^{M+\lambda_2} \cdots (n-1)^{M+\lambda_{n-1}} n^{\alpha M+\nu} (n-1)^{\mu_1+\dots+\mu_{n-1}} \cdots 2^{\mu_1+\mu_2} 1^{\mu_1}),$$

where $\alpha = 1$ for $\mathfrak{g}_0$ of type C_n and $\alpha = 2$ for $\mathfrak{g}_0$ of type B_n. By direct calculation, one finds that $\Phi(P) = f_{\vec{a}}(u)$. Hence it is clear that $\overline{S}(\Phi(P)) = \Phi(\hat{P})$, where $\hat{P}$ is the doubled $\pm$-diagram of P.

Now we show the claim for the general case by induction on N, where N is the number of $+$ in P at height lower than n. If $N = 0$, the statement was shown above. Suppose $N > 0$. Let h be the height of the lowest $+$ in P, and P' the $\pm$-diagram obtained by replacing the leftmost $+$ at height h with a box. Let t be the rightmost column of $\Phi(P)$ such that it contains $\overline{h+1}$ or does not contain $h+1$. Then $\Phi(P')$ differs from $\Phi(P)$ only in the column at the position where t is situated and there are three cases, except when $N = 1$ and P has a spin column with $-$. (This exceptional case can be checked easily and is not considered later.) Let t' be

the corresponding column in P'. Then the three cases are as follows:

$$\text{(i)} \quad t = \begin{array}{|c|}\hline k \\ \hline \vdots \\ \hline h+2 \\ \hline h \\ \hline \vdots \\ \hline 1 \\ \hline \end{array}, \qquad t' = \begin{array}{|c|}\hline k \\ \hline \vdots \\ \hline h+2 \\ \hline h+1 \\ \hline \vdots \\ \hline 2 \\ \hline \end{array}$$

$$\text{(ii)} \quad t = \begin{array}{|c|}\hline \overline{k'} \\ \hline k \\ \hline \vdots \\ \hline h+2 \\ \hline h \\ \hline \vdots \\ \hline 1 \\ \hline \end{array}, \qquad t' = \begin{array}{|c|}\hline \overline{k'} \\ \hline k \\ \hline \vdots \\ \hline h+2 \\ \hline h+1 \\ \hline \vdots \\ \hline 2 \\ \hline \end{array}$$

$$\text{(iii)} \quad t = \begin{array}{|c|}\hline \overline{h+1} \\ \hline k \\ \hline \vdots \\ \hline 2 \\ \hline \end{array}, \qquad t' = \begin{array}{|c|}\hline \overline{1} \\ \hline k \\ \hline \vdots \\ \hline 2 \\ \hline \end{array}.$$

Here $k, k' \geq h+1$, and $k = h+1$ case in (i) or (ii) means that there is no letter from $h+2$ to k. To the left of t we have columns of type

$$\begin{array}{|c|}\hline k \\ \hline \vdots \\ \hline h'+2 \\ \hline h' \\ \hline \vdots \\ \hline 1 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|}\hline \overline{k'} \\ \hline k \\ \hline \vdots \\ \hline h'+2 \\ \hline h' \\ \hline \vdots \\ \hline 1 \\ \hline \end{array}$$

where $h' \geq h, k \geq h'+1, k' \geq h+1$, and to the right we have

$$\begin{array}{|c|}\hline k \\ \hline \vdots \\ \hline 2 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|}\hline \overline{1'} \\ \hline k \\ \hline \vdots \\ \hline 2 \\ \hline \end{array}.$$

Let t^* be the leftmost column of $\Phi(P)$ that is to the right of t and does not contain $h+1$. Let n_j $(2 \leq j \leq h)$ be the number of boxes with letter j that is weakly right of t^*. Setting

$$\vec{a} = (12^{n_2+1} \cdots (h-1)^{n_{h-1}+1} h^{n_h+1}),$$
$$\vec{c} = (h^{n_h}(h-1)^{n_{h-1}} \cdots 2^{n_2}),$$

we have $\Phi(P') = e_{\vec{c}} f_{\vec{a}} \Phi(P)$ in all three cases (i),(ii),(iii).

Next we consider the doubled $\pm$-diagrams $\hat{P}$ and $\hat{P}'$ corresponding to P and P'. Then $\Phi(\hat{P}')$ differs from $\Phi(\hat{P})$ by two columns as in (i') and one column as in (ii'), where

$$\text{(i')} \quad \Phi(\hat{P}) : \begin{array}{|c|c|} \hline k & k \\ \hline \vdots & \vdots \\ \hline h+2 & h+2 \\ \hline h & h \\ \hline \vdots & \vdots \\ \hline 1 & 1 \\ \hline \end{array}, \qquad \Phi(\hat{P}') : \begin{array}{|c|c|} \hline k & k \\ \hline \vdots & \vdots \\ \hline h+2 & h+2 \\ \hline h+1 & h+1 \\ \hline \vdots & \vdots \\ \hline 2 & 2 \\ \hline \end{array}$$

$$\text{(ii')} \quad \Phi(\hat{P}) : \begin{array}{|c|} \hline \overline{h+1} \\ \hline k \\ \hline \vdots \\ \hline h+2 \\ \hline h \\ \hline \vdots \\ \hline 1 \\ \hline \end{array}, \qquad \Phi(\hat{P}') : \begin{array}{|c|} \hline \overline{1} \\ \hline k \\ \hline \vdots \\ \hline h+2 \\ \hline h+1 \\ \hline \vdots \\ \hline 2 \\ \hline \end{array}.$$

Let $\hat{t}^*$ be the leftmost column of $\Phi(\hat{P})$ that is to the right of the above one and does not contain $h+1$. The number of boxes with letter j that is weakly right to $\hat{t}^*$ is given by $2c_j$. Calculating carefully we obtain $\Phi(\hat{P}') = e_{\vec{c'}} f_{\vec{a'}} \Phi(\hat{P})$, where $\vec{a'}$ and $\vec{c'}$ are sequences obtained from $\vec{a}, \vec{c}$ by repeating each letter twice, namely, if $\vec{a} = (a_1, a_2, \dots, a_m)$, then $\vec{a'} = (a_1^2, a_2^2, \dots, a_m^2)$. Therefore, we have

$$S(\Phi(P)) = e_{\overleftarrow{a'}} f_{\overleftarrow{c'}} S(\Phi(P')) = e_{\overleftarrow{a'}} f_{\overleftarrow{c'}} \Phi(\hat{P}') = \Phi(\hat{P}),$$

by using the induction hypothesis. The proof is finished. $\square$

4. Combinatorial KR crystals

In this section we define the combinatorial KR crystals $V^{r,s}$. In Sections 4.1 and 4.2 we review types $A_{n-1}^{(1)}$ and $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$, respectively. In Section 4.3 we use the folding technique to construct $V^{r,s}$ of type $C_n^{(1)}$ as a "virtual" crystal inside $V^{r,s}$ of type $A_{2n+1}^{(2)}$, and use this in Section 4.4 to define the combinatorial KR crystals of type $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$ using the similarity method of Kashiwara [20]. The construction for exceptional nodes will be given in Section 6. In Section 4.5 we derive some properties of φ_0 that will be used later.

4.1. **Combinatorial KR crystals of type $A_{n-1}^{(1)}$.** The existence of the KR crystal $B^{r,s}$ of type $A_{n-1}^{(1)}$ was shown in [18]. A combinatorial description of this crystal was provided by Shimozono [33]. As a $\{1, 2, \dots, n-1\}$-crystal

$$B^{r,s} \cong B(s\Lambda_r) \tag{4.1}$$

which as a set are all semi-standard Young tableaux of rectangular shape (s^r) over the alphabet $1 \prec 2 \prec \cdots \prec n$. The Dynkin diagram of $A_{n-1}^{(1)}$ has a cyclic

automorphism $i \mapsto i+1 \pmod{n}$. The action of the affine crystal operators f_0 and e_0 is given by

$$f_0 = \mathrm{pr}^{-1} \circ f_1 \circ \mathrm{pr} \qquad \text{and} \qquad e_0 = \mathrm{pr}^{-1} \circ e_1 \circ \mathrm{pr}, \tag{4.2}$$

where pr is Schützenberger's promotion operator [32], which is the analogue of the cyclic Dynkin diagram automorphism on the level of crystals. On a rectangular tableau $b \in B^{r,s}$, $\mathrm{pr}(b)$ is obtained from b by removing all letters n, adding one to each letter in the remaining tableau, using jeu-de-taquin to slide all letters up, and finally filling the holes with 1s.

Theorem 4.1. [33, Section 3.3] *For type $A_{n-1}^{(1)}$, the crystal $B^{r,s}$ decomposes as in* (4.1) *as a $\{1, 2, \ldots, n-1\}$-crystal with the affine crystal action as given in* (4.2).

4.2. **Combinatorial KR crystals of type $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$.** Let $\mathfrak{g}$ be of type $D_n^{(1)}$, $B_n^{(1)}$, or $A_{2n-1}^{(2)}$ with the underlying finite Lie algebra $\mathfrak{g}_0$ of type $X_n = D_n, B_n$, or C_n, respectively. In this section we review the combinatorial model for KR crystals $B^{r,s}$ of type $\mathfrak{g}$ as given in [28, 31], where $r \neq n-1, n$ for $D_n^{(1)}$. The case $r = n$ for $B_n^{(1)}$ treated in Lemma 4.2 in this section is new. The cases $r = n-1, n$ for $D_n^{(1)}$ are treated in Section 6.2. The crystals for type $A_{2n-1}^{(2)}$ will be used in Section 4.3 in order to define the combinatorial KR crystals for type $C_n^{(1)}$ using the folding technique.

The Dynkin diagrams of type $D_n^{(1)}, B_n^{(1)}$, and $A_{2n-1}^{(2)}$ all have an automorphism interchanging nodes 0 and 1. The analogue σ of this automorphism on the level of crystals exists. By construction the automorphism σ commutes with f_i and e_i for $i = 2, 3, \ldots, n$. Hence it suffices to define σ on X_{n-1} highest weight elements where X_{n-1} is the subalgebra whose Dynkin diagram is obtained from that of X_n by removing node 1. Because of the bijection Φ between $\pm$-diagrams and X_{n-1}-highest weight elements as described in Section 3.2, it suffices to define the map on $\pm$-diagrams.

For the following description of the map $\mathfrak{S}$, we assume that $1 \le r \le n$ for $A_{2n-1}^{(2)}$, $1 \le r \le n-1$ for $B_n^{(1)}$ and $1 \le r \le n-2$ for $D_n^{(1)}$. Let P be a $\pm$-diagram of shape Λ/λ. Let $c_i = c_i(\lambda)$ be the number of columns of height i in λ for all $1 \le i < r$ with $c_0 = s - \lambda_1$. If $i \equiv r-1 \pmod{2}$, then in P, above each column of λ of height i, there must be a $+$ or a $-$. Interchange the number of such $+$ and $-$ symbols. If $i \equiv r \pmod{2}$, then in P, above each column of λ of height i, either there are no signs or a $\mp$ pair. Suppose there are p_i $\mp$ pairs above the columns of height i. Change this to $(c_i - p_i)$ $\mp$ pairs. The result is $\mathfrak{S}(P)$, which has the same inner shape λ as P but a possibly different outer shape.

Definition 4.1. Let $1 \le r \le n$ for $A_{2n-1}^{(2)}$, $1 \le r \le n-1$ for $B_n^{(1)}$ and $1 \le r \le n-2$ for $D_n^{(1)}$. Let $b \in B^{r,s}$ and $e_{\vec{\mathbf{a}}} := e_{a_1} e_{a_2} \cdots e_{a_\ell}$ be such that $e_{\vec{\mathbf{a}}}(b)$ is a X_{n-1} highest weight crystal element. Define $f_{\overleftarrow{\mathbf{a}}} := f_{a_\ell} f_{a_{\ell-1}} \cdots f_{a_1}$. Then

$$\sigma(b) := f_{\overleftarrow{\mathbf{a}}} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ e_{\vec{\mathbf{a}}}(b). \tag{4.3}$$

As an X_n-crystal, $B^{r,s}$ decomposes into the following irreducible components

$$B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda), \tag{4.4}$$

with the exception of $r = n-1, n$ for $D_n^{(1)}$. Here $B(\Lambda)$ is the X_n-crystal of highest weight Λ and the sum runs over all dominant weights Λ that can be obtained from $s\Lambda_r$ by the removal of vertical dominoes, where Λ_i are the fundamental weights of X_n. The decomposition (4.4) also holds as a $\{0, 2, 3, \ldots, n\}$-crystal.

Definition 4.2. Let $V^{r,s}$ for $s \geq 1$ and $1 \leq r \leq n$ for type $A_{2n-1}^{(2)}$, $1 \leq r \leq n-1$ for type $B_n^{(1)}$, and $1 \leq r \leq n-2$ for type $D_n^{(1)}$ be defined as follows. As an X_n crystal

$$V^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda), \tag{4.5}$$

where the sum runs over all dominant weights Λ that can be obtained from $s\Lambda_r$ by the removal of vertical dominoes. The affine crystal operators e_0 and f_0 are defined as

$$\begin{aligned} f_0 &= \sigma \circ f_1 \circ \sigma, \\ e_0 &= \sigma \circ e_1 \circ \sigma. \end{aligned} \tag{4.6}$$

Next we give a definition of $V^{n,s}$ for type $B_n^{(1)}$ using Kashiwara's similarity technique [20]. Note that the classical decomposition (4.4) is still valid in this case.

Lemma 4.2. *Let $B^{n,s}$ be the $A_{2n-1}^{(2)}$-KR crystal. Then there exists a regular $B_n^{(1)}$-crystal $V^{n,s}$ and a unique injective map $S : V^{n,s} \to B^{n,s}$ such that*

$$S(e_i b) = e_i^{m_i} S(b), \quad S(f_i b) = f_i^{m_i} S(b) \quad \text{for } i \in I,$$

where $(m_i)_{0 \leq i \leq n} = (2, 2, \ldots, 2, 1)$. Furthermore, $V^{n,s}$ decomposes as (4.5) as a $\{1, 2, \ldots, n\}$-crystal and as a $\{0, 2, \ldots, n\}$-crystal.

Proof. As a B_n-crystal define

$$V^{n,s} = \bigoplus_{\mathbf{k}} B(k_\iota \Lambda_\iota + k_{\iota+2}\Lambda_{\iota+2} + \cdots + k_n \Lambda_n)$$

where the sum is over $\mathbf{k} \in \{(k_\iota, k_{\iota+2}, \ldots, k_n) \mid 2(k_\iota + k_{\iota+2} + \cdots + k_{n-2}) + k_n = s\}$ with $\iota \equiv n \pmod 2$, $\iota = 0$ or 1, and $B(\Lambda)$ is the highest weight B_n-crystal of highest weight Λ, where we identify $\Lambda_0 = 0$. Then by [20, Theorem 5.1] with $\xi = \mathrm{id}$ and $(m_i)_{i \in \{1,2,\ldots,n\}}$, one realizes that there exists a unique injective map

$$\overline{S} : V^{n,s} \longrightarrow \bigoplus_{\mathbf{k}} B_{C_n}(2k_\iota \Lambda_\iota + 2k_{\iota+2}\Lambda_{\iota+2} + \cdots + 2k_{n-2}\Lambda_{n-2} + k_n \Lambda_n)$$

such that $\overline{S}(e_i b) = e_i^{m_i} \overline{S}(b)$ and $\overline{S}(f_i b) = f_i^{m_i} \overline{S}(b)$ for $i \in \{1, 2, \ldots, n\}$. Note that the RHS is contained in $B^{n,s}$. Since σ is defined on $B^{n,s}$, to finish the proof it suffices to show that $\overline{S}(V^{n,s})$ is closed under σ. This is clear from Lemma 3.5.

By construction, $V^{n,s}$ decomposes as (4.5) as a $\{1, 2, \ldots, n\}$-crystal. Since e_0 is defined using σ, the same decomposition holds as a $\{0, 2, \ldots, n\}$-crystal. □

Let us now state [31, Lemma 5.1] since we will need it several times in the sequel. The lemma describes the action of e_1 on $\{3, 4, \ldots, n\}$ highest weight elements which are in one-to-one correspondence with pairs of $\pm$-diagrams (P, p), where the inner shape of P is the outer shape of p.

The operator e_1 either changes a 2 into a 1 or a $\overline{1}$ into a $\overline{2}$ in b corresponding to the pair of $\pm$-diagrams $\Psi(P, p)$. On the level of (P, p) this means that either a $+$ from p transfers to P, or a $-$ moves from P to p. To describe the precise action of e_1 on (P, p) perform the following algorithm:

(1) Successively run through all $+$ in p from left to right and, if possible, pair it with the leftmost yet unpaired $+$ in P weakly to the left of it.
(2) Successively run through all $-$ in p from left to right and, if possible, pair it with the rightmost yet unpaired $-$ in P weakly to the left.
(3) Successively run through all yet unpaired $+$ in p from left to right and, if possible, pair it with the leftmost yet unpaired $-$ in p.

Lemma 4.3. [31, Lemma 5.1] *If there is an unpaired $+$ in p, e_1 moves the rightmost unpaired $+$ in p to P. Else, if there is an unpaired $-$ in P, e_1 moves the leftmost unpaired $-$ in P to p. Else e_1 annihilates (P, p).*

4.3. **Combinatorial KR crystals for $C_n^{(1)}$.** In this section we define a combinatorial crystal $V^{r,s}$ of type $C_n^{(1)}$. For $r \neq n$, this combinatorial crystal is isomorphic to the KR crystal $B^{r,s}$ (see Theorem 5.7). The KR crystal for $r = n$ is treated in Section 6.1. We realize $V^{r,s}$ as a "virtual" crystal inside the ambient crystal $\hat{V}^{r,s} = B^{r,s}_{A^{(2)}_{2n+1}}$ of type $A^{(2)}_{2n+1}$. Let $I = \{0, 1, \ldots, n\}$ be the index set for the Dynkin diagram of type $C_n^{(1)}$ and $\hat{I} = \{0, 1, \ldots, n+1\}$ be the index set of the Dynkin diagram of type $A^{(2)}_{2n+1}$. We denote the crystal operators of $V^{r,s}$ by e_i and f_i, and the crystal operators of the ambient crystal $\hat{V}^{r,s}$ by $\hat{e}_i$ and $\hat{f}_i$. In this case

$$e_i = \begin{cases} \hat{e}_0 \hat{e}_1 & \text{if } i = 0 \\ \hat{e}_{i+1} & \text{if } 1 \le i \le n \end{cases} \quad \text{and} \quad f_i = \begin{cases} \hat{f}_0 \hat{f}_1 & \text{if } i = 0 \\ \hat{f}_{i+1} & \text{if } 1 \le i \le n. \end{cases} \tag{4.7}$$

The analogue of the Dynkin diagram automorphism of $A^{(2)}_{2n+1}$, which interchanges nodes 0 and 1, on the level of the crystal $B^{r,s}_{A^{(2)}_{2n+1}}$ is denoted by σ.

Definition 4.3. Define $V^{r,s}$ to be the subset of elements $b \in \hat{V}^{r,s}$ that are invariant under σ, namely $\sigma(b) = b$, together with the operators e_i and f_i of (4.7).

Lemma 4.4. *$V^{r,s}$ is closed under the operators e_i and f_i for $i \in I$.*

Proof. For $i \in \hat{I} \setminus \{0, 1\}$, the operators $\hat{e}_i$ and $\hat{f}_i$ commute with σ. Hence for $b \in V^{r,s}$ such that $e_i(b) \neq \emptyset$ we have

$$\sigma \circ e_i(b) = e_i \circ \sigma(b) = e_i(b) \quad \text{for } i \in I \setminus \{0\}$$

and similarly for f_i. Note also that $\hat{e}_0 = \sigma \circ \hat{e}_1 \circ \sigma$ and $\hat{f}_0 = \sigma \circ \hat{f}_1 \circ \sigma$. Hence for an element $b \in V^{r,s}$ such that $e_0(b) \neq \emptyset$ we have

$$\begin{aligned} \sigma \circ e_0(b) = \sigma \circ \hat{e}_0 \circ \hat{e}_1(b) = \sigma \circ \hat{e}_1 \circ \hat{e}_0(b) = \sigma \circ \hat{e}_1 \circ \sigma \circ \hat{e}_1 \circ \sigma(b) \\ = (\sigma \circ \hat{e}_1 \circ \sigma) \circ \hat{e}_1(b) = \hat{e}_0 \circ \hat{e}_1(b) = e_0(b) \end{aligned}$$

and similarly for f_0. □

Lemma 4.5. *As a $\{1, 2, \ldots, n\}$-crystal*

$$V^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda)$$

where the sum is over all Λ obtained from $s\Lambda_r$ by removing horizontal dominoes and $B(\Lambda)$ is a highest weight C_n-crystal of highest weight Λ.

Proof. As a $\{1,2,\dots,n\}$-crystal $V^{r,s}$ is isomorphic to the connected components of σ invariant elements of the $\{2,3,\dots,n+1\}$-crystal $\hat{V}^{r,s}$ of type $A^{(2)}_{2n+1}$. Since $\hat{e}_i$ and $\hat{f}_i$ commute with σ for $i \in \hat{I} \setminus \{0,1\}$, the crystal of σ invariant elements of the $\{2,3,\dots,n+1\}$-crystal $\hat{V}^{r,s}$ of type $A^{(2)}_{2n+1}$ is a disjoint union of C_n-crystals. The $\{2,3,\dots,n+1\}$-highest weight elements in $\hat{V}^{r,s}$ are in one-to-one correspondence with $\pm$-diagrams. Hence to find the decomposition we need to list all σ-invariant $\pm$-diagrams. Let P be a $\pm$-diagram for $\hat{V}^{r,s}$ of type $A^{(2)}_{2n+1}$. Let λ be the outer shape of P and Λ be the inner shape of P. For P to be σ-invariant, for any row i of Λ with $r-i \equiv 1 \pmod 2$, the cells above the cells in row i of Λ must contain the same number of $+$ and $-$ signs in P. Similarly, for rows $i<r$ in Λ with $r-i \equiv 0 \pmod 2$, the cells above the cells in row i of Λ must contain the same number of $\pm$ pairs as no additional cells in P. This shows in particular that the rows of Λ have the same parity as s, and hence Λ can be obtained from $s\Lambda_r$ by removal of horizontal dominoes. As a C_n-weight, the weight of the $\{2,3,\dots,n+1\}$- highest weight element corresponding to P is precisely Λ. This proves the claim. □

Lemma 4.6. *As a $\{0,1,\dots,n-1\}$-crystal*

$$V^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda)$$

where the sum is over all Λ obtained from $s\Lambda_r$ by removing horizontal dominoes and $B(\Lambda)$ is a highest weight C_n-crystal of highest weight Λ.

Proof. Proposition 3.2.1 of [25] applies to the situation, where $\mathfrak{g} = A^{(2)}_{2n+1}$, $\hat{\mathfrak{g}} = C^{(1)}_n$, ω is the Dynkin diagram automorphism interchanging nodes 0 and 1 (this Proposition applies even though $\omega(0) \neq 0$ in this case), and $B = B^{r,s}$ of type $A^{(2)}_{2n+1}$. Then the Proposition ensures that the fixed point subset B^ω (or $V^{r,s}$ in our notation) with crystal operators as in (4.7) is a regular $U'_q(\hat{\mathfrak{g}})$-crystal.

The Weyl group of type C_n contains an element that maps $\Lambda_j - \Lambda_0$ to $\Lambda_{n-j} - \Lambda_n$ for all $1 \le j \le n$. Under this map, the highest weight elements in $V^{r,s}$ as a $\{1,2,\dots,n\}$-crystal map to highest weight elements in $V^{r,s}$ as a $\{0,1,\dots,n-1\}$-crystal. Hence the decomposition of the claim follows from Lemma 4.5. □

Lemma 4.7.

(1) *Let $b \in V^{r,s}$ be a $\{2,\dots,n\}$-highest element corresponding to the $\pm$-diagram P. Suppose the number of $\mp, +, -$ or $\cdot$ on the columns of the inner shape of the same height smaller than r is always even. Then the $\pm$-diagram corresponding to $e_0(b)$ also has the same property.*

(2) *Let Λ, Λ' be weights appearing in the decomposition of $V^{r,s}$ as in Lemma 4.5 such that Λ' is obtained from Λ by adding a horizontal domino. Then there exists a $\{2,\dots,n\}$-highest weight element $b \in B(\Lambda)$ such that $e_0(b) \in B(\Lambda')$.*

Proof. Let us first prove (1). Inside the ambient crystal $\hat{V}^{r,s}$, the element b is a $\{3,\dots,n+1\}$ highest weight vector. It corresponds to a pair of $\pm$-diagrams (P', P), where P is the same as in the statement of the lemma and P' is the $\pm$-diagram corresponding to the highest weight vector in the component of b obtained as described in the proof of Lemma 4.5. The crystal operator e_0 in $V^{r,s}$ corresponds to $\hat{e}_1\hat{e}_0 = \hat{e}_1\sigma\hat{e}_1\sigma = \hat{e}_1\sigma\hat{e}_1$ in $\hat{V}^{r,s}$, where in the last step we used that $b \in V^{r,s}$ is invariant under σ. By Lemma 4.3, $\hat{e}_1$ either moves a $+$ from P to P', or a $-$

from P' to P. In this case σ on $\hat{e}_1(P')$ either has one extra $-$, or one fewer $+$, respectively. Since by Lemma 4.4, $V^{r,s}$ is closed under e_0 and the number of $+$ and $-$ is balanced in $\pm$-diagrams corresponding to $\{2,\dots,n\}$-highest weight vectors, $e_0(P')$ must have this property. This implies that $e_0(P)$ either has two less $+$ in the same row or two extra $-$ in the same row (since the shape can only change by horizontal dominoes). This proves the claim.

For the proof of (2), let P' be as before with inner shape Λ. Suppose Λ' has two more boxes than Λ in row j. Let P be the $\pm$-diagram with outer shape Λ corresponding to the partition $\lambda = (\lambda_1,\dots,\lambda_n)$ such that row n is filled with $-$ and each row λ_i for $n > i \neq j-1$ is filled with $(\lambda_i - \lambda_{i+1})/2$ minus signs. Then by the previous description of e_0 using Lemma 4.3, the $\pm$-diagram corresponding to $e_0(b)$ has outer shape Λ' and two more minus signs in row j. □

4.4. **Combinatorial KR crystals for $A_{2n}^{(2)}, D_{n+1}^{(2)}$.** We use the similarity technique of Kashiwara developed in [20] to define combinatorial models of KR crystals for $A_{2n}^{(2)}, D_{n+1}^{(2)}$ for nonexceptional nodes from those for $C_n^{(1)}$. In this subsection $V^{r,s}$ stands for the $C_n^{(1)}$-crystal defined in the previous subsection. For types $A_{2n}^{(2)}, D_{n+1}^{(2)}$ define positive integers m_i for $i \in I$ as follows:

$$(m_0, m_1, \dots, m_{n-1}, m_n) = \begin{cases} (1,2,\dots,2,2) & \text{for } A_{2n}^{(2)}, \\ (1,2,\dots,2,1) & \text{for } D_{n+1}^{(2)}. \end{cases} \tag{4.8}$$

The next theorem ensures the existence of crystals $V_{\mathfrak{g}}^{r,s}$ for type $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$. They are constructed from type $C_n^{(1)}$ crystals by an affine extension of $\overline{S}$ of Section 3.3.

Theorem 4.8. *For $1 \le r \le n$ for $\mathfrak{g} = A_{2n}^{(2)}$, $1 \le r < n$ for $\mathfrak{g} = D_{n+1}^{(2)}$ and $s \ge 1$, there exists a $\mathfrak{g}$-crystal $V_{\mathfrak{g}}^{r,s}$ and a unique injective map $S : V_{\mathfrak{g}}^{r,s} \longrightarrow V^{r,2s}$ such that*

$$S(e_i b) = e_i^{m_i} S(b), \quad S(f_i b) = f_i^{m_i} S(b) \qquad \text{for } i \in I.$$

Proof. Let Λ be a dominant integral weight of $\mathfrak{g}_0$ where $\mathfrak{g}_0$ is C_n, B_n for $\mathfrak{g} = A_{2n}^{(2)}, D_{n+1}^{(2)}$, respectively. We assume if $\mathfrak{g} = D_{n+1}^{(2)}$, the coefficient of the n-th fundamental weight of Λ is zero. As before let $B(\Lambda)$ (resp. $B_{\mathfrak{g}_0}(\Lambda)$) be the C_n (resp. $\mathfrak{g}_0$)-crystal of the highest weight module of highest weight Λ. We saw that by [20, Theorem 5.1], there exists a unique injective map $\overline{S} : B_{\mathfrak{g}_0}(\Lambda) \to B(2\Lambda)$ such that $\overline{S}(e_i b) = e_i^{m_i}\overline{S}(b)$ and $\overline{S}(f_i b) = f_i^{m_i}\overline{S}(b)$ for $i \in I_0 = \{1,2,\dots,n\}$.

Now define $V_{\mathfrak{g}}^{r,s}$, as a $\mathfrak{g}_0$-crystal, by

$$V_{\mathfrak{g}}^{r,s} = \bigoplus_{\Lambda} B_{\mathfrak{g}_0}(\Lambda), \tag{4.9}$$

where the sum is over all Λ obtained from $s\Lambda_r$ by removing nodes. Then from the above we have an injective map $S : V_{\mathfrak{g}}^{r,s} \to V^{r,2s}$. To make $V_{\mathfrak{g}}^{r,s}$ a $\mathfrak{g}$-crystal, it suffices to introduce the 0-action on $V_{\mathfrak{g}}^{r,s}$. Take an element $b \in V_{\mathfrak{g}}^{r,s}$. One can assume b is $\{2,\dots,n\}$-highest, hence b corresponds to a $\pm$-diagram P. Then $S(b)$ corresponds to a $\pm$-diagram $\tilde{P}$, where the number of $\mp, +, -$ or $\cdot$ on the columns of the inner shape of the same height is doubled from that of P by Lemma 3.5. By Lemma 4.7, $e_0 S(b)$ also has the same property, so there exists a $b' \in V_{\mathfrak{g}}^{r,s}$ such that $S(b') = e_0 S(b)$. Then one can define $e_0 b = b'$ on $V_{\mathfrak{g}}^{r,s}$ and we have $S(e_0 b) = e_0 S(b)$ by definition. The case f_0 is similar. □

Lemma 4.9. *For $1 \le r \le n$ for $\mathfrak{g} = A_{2n}^{(2)}$, $1 \le r < n$ for $\mathfrak{g} = D_{n+1}^{(2)}$ and $s \ge 1$,*

(1) *as a $\{1, 2, \dots, n\}$-crystal*
$$V_{\mathfrak{g}}^{r,s} \simeq \bigoplus_{\Lambda} B_{\mathfrak{g}_0}(\Lambda)$$
where the sum is over all Λ obtained from an $r \times s$ rectangle by removing boxes and $B_{\mathfrak{g}_0}(\Lambda)$ is a highest weight $\mathfrak{g}_0$-crystal of highest weight Λ;

(2) *as a $\{0, 1, \dots, n-1\}$-crystal*
$$V_{\mathfrak{g}}^{r,s} \simeq \bigoplus_{\Lambda} B_{B_n}(\Lambda)$$
where the sum is over all Λ obtained from an $r \times s$ rectangle by removing horizontal dominoes for $\mathfrak{g} = A_{2n}^{(2)}$ and boxes for $\mathfrak{g} = D_{n+1}^{(2)}$, and $B_{B_n}(\Lambda)$ is a highest weight B_n-crystal of highest weight Λ.

Proof. Decomposition (1) is true by construction, see the proof of Theorem 4.8 and in particular (4.9). Once (1) is established, the decomposition as a $\{0, 1, \dots, n-1\}$-crystal is unique. See e.g. Section 6.2 of [8] for this result. □

4.5. **Some properties of φ_0.** In this section $V^{r,s}$ is the crystal of type $C_n^{(1)}$, $A_{2n}^{(2)}$ or $D_{n+1}^{(2)}$ as defined in Sections 4.3 and 4.4.

Proposition 4.10. *Let $b \in V^{r,s}$ be a $\{2, \dots, n\}$-highest weight element and P the corresponding $\pm$-diagram.*

(1) *Suppose that P contains only empty columns $\cdot$ or ε ($\varepsilon = +$ or $-$). Let λ be the inner shape of P. We move through the columns of λ with height less than r from left to right. (We consider there are $s - \text{width}(b)$ columns of height 0.) Delete all $(\cdot, \varepsilon)$ pairs successively until we have some ε's followed by some $\cdot$'s. Then, setting $m_0 = 2$ for $C_n^{(1)}$ and $m_0 = 1$ for $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$:*
   (a) *If $\varepsilon = +$, $\varphi_0(b)$ is the number of the remaining $\cdot$ divided by m_0.*
   (b) *If $\varepsilon = -$, $\varphi_0(b)$ is the number of the remaining $-$ divided by m_0 plus $(s - c_r)/m_0$, where c_r is the number of columns in λ of height r.*

(2) *Suppose that the outer shape of P has at least one column of height strictly less than $n-1$, but P does not contain any $+$ in columns of height strictly less than $n-1$. Then $\varphi_0(b) > 0$.*

Proof. We first treat case (1) for type $C_n^{(1)}$. As in the proof of Lemma 4.7, inside the ambient crystal $\hat{V}^{r,s}$, the element b is a $\{3, \dots, n+1\}$-highest weight vector and corresponds to a pair of $\pm$-diagrams (P', P), where P is the same as in the statement of the lemma and P' is the $\pm$-diagram corresponding to the highest weight vector in the component of b which is invariant under σ. Since f_0 corresponds to $\hat{f}_0 \hat{f}_1$ in $\hat{V}^{r,s}$, and $\hat{f}_0$ and $\hat{f}_1$ commute, it suffices to determine $\widehat{\varphi}_1$ of the element corresponding to the pair of $\pm$-diagrams (P', P) to obtain $\varphi_0(b)$. We again employ Lemma 4.3.

First suppose that $\varepsilon = +$. Recall that this means that P does not contain any $-$. After bracketing the $+$ in P with the $+$ in P', φ_0 is determined by the number of unbracketed $+$ in P' by Lemma 4.3. Recall that if b sits in the C_n component of highest weight Λ, then P' is obtained by adding above each column in Λ either the same number of $+$ and $-$, or the same number of $\mp$ and $\cdot$. With this, it is not hard to check that the rule stated in the lemma coincides with the bracketing arguments of Lemma 4.3.

Next suppose that $\varepsilon = -$. In this case P only contains $-$ and after bracketing the $-$ in P with the $-$ in P', φ_0 is obtained by the number of $+$ in P' (which equals $(s-c_r)/2$) plus the number of unbracketed $-$ in P (which by similar arguments as in the case $\varepsilon = +$ is equal to half of the number of unbracked $-$ by the rule of the lemma).

The proof for type $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ is now immediate by Lemma 3.5.

For case (2), we consider again the pair (P', P) as before. Since there is at least one column of height strictly less than $n-1$ and there are no $+$s in P in columns of height strictly less than $n-1$, we conclude by Lemma 4.3 that there is at least one unbracketed $+$ in P' and hence $\varphi_0(b) > 0$. □

5. Uniqueness theorem

Recall that $B^{r,s}$ denotes the KR crystal and $V^{r,s}$ denotes the combinatorial crystals defined in Section 4. In this section we show that given certain decompositions of $B^{r,s}$ into subcrystals, $B^{r,s}$ is uniquely determined, thereby proving that $V^{r,s} \cong B^{r,s}$. In Section 5.1 we review the proof of [28] for types $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$ and extend it to $r = n$ of type $B_n^{(1)}$. For type $C_n^{(1)}$, $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$ the uniqueness proof is given in Theorem 5.7 in Section 5.2.

5.1. **Uniqueness for types $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$.** Let $V^{r,s}$ be the combinatorial crystal of type $D_n^{(1)}$, $B_n^{(1)}$, or $A_{2n-1}^{(2)}$ as defined in Section 4.2. By definition of $V^{r,s}$ we know that for $1 \le r \le n-2$ for type $D_n^{(1)}$ and for $1 \le r \le n$ for types $B_n^{(1)}$ and $A_{2n-1}^{(2)}$ there exist the isomorphisms

$$\begin{aligned} \Psi_0 : V^{r,s} \simeq B^{r,s} \qquad & \text{as an isomorphism of } \{1,2,\ldots,n\}\text{-crystals},\\ \Psi_1 : V^{r,s} \simeq B^{r,s} \qquad & \text{as an isomorphism of } \{0,2,\ldots,n\}\text{-crystals}. \end{aligned}$$

Theorem 5.1. *Let $s \ge 1$ and $1 \le r \le n$ for type $A_{2n-1}^{(2)}$, $1 \le r \le n-1$ for type $B_n^{(1)}$, and $1 \le r \le n-2$ for type $D_n^{(1)}$. Then $\Psi_0(b) = \Psi_1(b)$ for all $b \in V^{r,s}$, and hence there exists a unique I-crystal isomorphism $\Psi : V^{r,s} \cong B^{r,s}$.*

We review the proof of [28, Proposition 6.1] of this theorem here, with special attention to the the case $r = n$ for type $A_{2n-1}^{(2)}$. This result is used in Theorem 5.3 below for the uniqueness proof for $r = n$ of type $B_n^{(1)}$. Let us first recall a Lemma and Remark that is needed for the proof of Theorem 5.1.

Lemma 5.2. *Let $b \in V^{r,s}$ be an X_{n-2}-highest weight vector corresponding to the tuple of $\pm$-diagrams (P,p) where all columns of height less than $n-1$ in p are empty and all columns of height $n-1$ in p contain $-$. Assume that $\varepsilon_0(b), \varepsilon_1(b) > 0$. Then* $\mathrm{inner}(b)$ *is strictly contained in* $\mathrm{inner}(e_0(b))$, $\mathrm{inner}(e_1(b))$, *and* $\mathrm{inner}(e_0e_1(b))$.

Proof. By assumption p does not contain any $+$ and e_1 is defined. Hence by Lemma 4.3, e_1 moves a $-$ in P to p. This implies that the inner shape of b is strictly contained in the inner shape of $e_1(b)$.

The involution σ does not change the inner shape of b (only the outer shape). By the same arguments as before, the inner shape of b is strictly contained in the inner shape of $e_1\sigma(b)$. Since σ does not change the inner shape, this is still true for $e_0(b) = \sigma e_1 \sigma(b)$.

Now let us consider $e_0e_1(b)$. For the change in inner shape we only need to consider $e_1\sigma e_1(b)$. By the same arguments as before, e_1 moves a $-$ from P to p and σ does not change the inner shape. The next e_1 will move another $-$ in $\sigma e_1(b)$ to p. Hence p will have grown by two $-$, so that the inner shape of $e_1\sigma e_1(b)$ is increased by two boxes. □

Remark 5.1. Note that Ψ_0 and Ψ_1 preserve weights, that is, $\mathrm{wt}\,(b) = \mathrm{wt}\,(\Psi_0(b)) = \mathrm{wt}\,(\Psi_1(b))$ for all $b \in V^{r,s}$. This is due to the fact that if all but one coefficient m_j are known for a weight $\Lambda = \sum_{j=0}^n m_j\Lambda_j$, then the missing m_j is also determined by the condition that the weights in KR crystals are of level zero.

Proof of Theorem 5.1. If $\Psi_0(b) = \Psi_1(b)$ for a b in a given X_{n-1}-component $\mathcal{C}$, then $\Psi_0(b') = \Psi_1(b')$ for all $b' \in \mathcal{C}$ since $e_i\Psi_0(b') = \Psi_0(e_ib')$ and $e_i\Psi_1(b') = \Psi_1(e_ib')$ for $i \in J = \{2, 3, \dots, n\}$. Hence it suffices to prove $\Psi_0(b) = \Psi_1(b)$ for only one element b in each X_{n-1}-component $\mathcal{C}$. We are going to establish the theorem for b corresponding to the pairs of $\pm$-diagrams (P, p) where all columns of p of height smaller than $n-1$ are empty and the columns of height $n-1$ are filled with $-$ (p can only contain columns of height $n-1$ for type $A_{2n-1}^{(2)}$ and $r = n$). Note that this is an X_{n-2}-highest weight vector, but not necessarily an X_{n-1}-highest weight vector.

We proceed by induction on inner(b) by containment. First suppose that both $\varepsilon_0(b), \varepsilon_1(b) > 0$. By Lemma 5.2, the inner shape of e_0e_1b, e_0b, and e_1b is bigger than the inner shape of b, so that by induction hypothesis $\Psi_0(e_0e_1b) = \Psi_1(e_0e_1b)$, $\Psi_0(e_0b) = \Psi_1(e_0b)$, and $\Psi_0(e_1b) = \Psi_1(e_1b)$. Therefore we obtain

$$\begin{aligned} e_0e_1\Psi_0(b) = e_0\Psi_0(e_1b) = e_0\Psi_1(e_1b) = \Psi_1(e_0e_1b) = \Psi_0(e_0e_1b)& \\ = e_1\Psi_0(e_0b) = e_1\Psi_1(e_0b) = e_1e_0\Psi_1(b).& \end{aligned}$$

This implies that $\Psi_0(b) = \Psi_1(b)$.

Next we need to consider the cases when $\varepsilon_0(b) = 0$ or $\varepsilon_1(b) = 0$, which comprises the base case of the induction. Let us first treat the case $\varepsilon_1(b) = 0$. Recall that all columns of p of height smaller than $n-1$ are empty and columns of height $n-1$ are filled with $-$. Hence it follows from the description of the action of e_1 of Lemma 4.3, that $\varepsilon_1(b) = 0$ if and only if all columns of height smaller than n in P are either empty or contain $+$ and the columns of height n contain either $+$ or $-$.

> **Claim.** $\Psi_0(b) = \Psi_1(b)$ for all b corresponding to the pair of $\pm$-diagrams (P, p) where all columns of height smaller than n in P are either empty or contain $+$, and all columns of p of height smaller than $n-1$ are empty and columns of height $n-1$ are filled with $-$.

The claim is proved by induction on $k + k_-$, where k is the number of empty columns in P of height strictly smaller than r and k_- is the number of $-$ in P. For $k + k_- = 0$ the claim is true by weight considerations. Now assume the claim is true for all $0 \le k' < k + k_-$ and we will establish the claim for $k + k_-$. Suppose that $\Psi_1(b) = \Psi_0(\tilde{b})$ where $\tilde{b} \neq b$. By weight considerations $\tilde{b}$ must correspond to a pair of $\pm$-diagrams $(\tilde{P}, p)$, where $\tilde{P}$ has the same number of columns containing only $+$ or $-$ as P and some of the empty columns of P of height h strictly smaller than r could be replaced by columns of height $h+2$ containing $\mp$. Denote by k_+ the number of columns of height strictly less than n in P containing $+$. Then

$$m := \varepsilon_0(b) = k_+ + k,$$

since under σ all empty columns in P become columns with $\mp$ and columns containing $+$ become columns with $-$. By Lemma 4.3, then e_1 acts on $(\mathfrak{S}(P), p)$ as often as there are minus signs in $\mathfrak{S}(P)$ of height less than n, which is $k_+ + k$. Set $\hat{b} = e_1^a \tilde{b}$, where $a > 0$ is the number of columns in $\tilde{P}$ containing $\mp$ plus the number of columns of $\tilde{P}$ of height strictly less than n containing $-$. If $(\hat{P}, \hat{p})$ denotes the tuple of $\pm$-diagrams associated to $\hat{b}$, then compared to $(\tilde{P}, p)$ all $-$ from the $\mp$ pairs in $\tilde{P}$ and all $-$ in columns of height less than n moved to p. Note that the induction variable for $\hat{P}$ is $k + k_- - a < k + k_-$, so that by induction hypothesis $\Psi_0(\hat{b}) = \Psi_1(\hat{b})$. Hence

$$\Psi_1(b) = \Psi_0(\tilde{b}) = \Psi_0(f_1^a \hat{b}) = f_1^a \Psi_0(\hat{b}) = f_1^a \Psi_1(\hat{b}). \tag{5.1}$$

Note that

$$\varepsilon_0(\hat{b}) = \varepsilon_0(\tilde{b}) = m - a < m.$$

Hence

$$\begin{aligned} & e_0^m \Psi_1(b) = \Psi_1(e_0^m b) \neq \emptyset \\ \text{but} \qquad & e_0^m f_1^a \Psi_1(\hat{b}) = f_1^a \Psi_1(e_0^m \hat{b}) = \emptyset \end{aligned}$$

which contradicts (5.1). This implies that we must have $\tilde{b} = b$ proving the claim.

The case $\varepsilon_0(b) = 0$ can be proven in a similar fashion to the case $\varepsilon_1(b) = 0$. Using the explicit action of $\mathfrak{S}$ on P and Lemma 4.3, it follows that $\varepsilon_0(b) = 0$ if and only if all columns of P of height strictly less than n contain either $-$ or $\mp$ pairs.

> **Claim.** $\Psi_0(b) = \Psi_1(b)$ for all b corresponding to the pair of $\pm$-diagrams (P, p) where all columns of P of height strictly less than n contain either $-$ or $\mp$ pairs, and all columns of p of height smaller than $n-1$ are empty and columns of height $n-1$ are filled with $-$.

By induction on the number of $\mp$ pairs plus the number of $+$ at height n in P, this claim can be proven similarly as before (using the fact that $\mathfrak{S}$ changes columns with $-$ into columns with $+$ and columns with $\mp$ pairs into empty columns). □

Theorem 5.3. *For type $B_n^{(1)}$ with $V^{n,s}$ as in Lemma 4.2, we have $V^{n,s} \cong B^{n,s}$.*

Proof. The proof follows in the same way as the proof of Theorem 5.1 for type $A_{2n-1}^{(2)}$, keeping in mind that $\pm$-diagrams in $V^{n,s}$ are characterized using Lemma 3.5. □

5.2. **Uniqueness for types $C_n^{(1)}$, $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$.** In this section $V^{r,s}$ is the combinatorial crystal of type $C_n^{(1)}$ of Section 4.3 or of type $A_{2n}^{(2)}$ or $D_{n+1}^{(2)}$ as defined in Section 4.4. By Lemmas 4.5, 4.6 and 4.9, $V^{r,s}$ is isomorphic to $B^{r,s}$ as an $\{1, 2, \ldots, n\}$-crystal and as an $\{0, 1, \ldots, n-1\}$-crystal. We define the isomorphisms

$$\begin{aligned} \Psi_0 : V^{r,s} \simeq B^{r,s} \qquad & \text{as an isomorphism of } \{1, 2, \ldots, n\}\text{-crystals}, \\ \Psi_n : V^{r,s} \simeq B^{r,s} \qquad & \text{as an isomorphism of } \{0, 1, \ldots, n-1\}\text{-crystals}. \end{aligned}$$

In this section we show in Theorem 5.7 that, given Ψ_0 and Ψ_n, there exists a unique $\{0, 1, \ldots, n\}$-crystals isomorphism $\Psi : V^{r,s} \simeq B^{r,s}$ (with the exception of $r = n$ for $C_n^{(1)}$ and $D_{n+1}^{(2)}$ which is treated in Section 6.1).

We first prepare three preliminary lemmas that are used in the proof.

Lemma 5.4. *Let $b \in V^{r,s}$ be a J-lowest weight element. If b contains a $\overline{p}$, with $p \neq 1$, then there exists a sequence $\vec{a}$ with letters in $\{2, \ldots, n-1\}$, such that $\varepsilon_n(e_{\vec{a}} b) > 0$.*

Proof. By Lemma 3.1 the form of b is known. Let ℓ be the rightmost column, that contains a barred letter $\overline{p}$ which is not $\overline{1}$. If there is no unbarred letter in column ℓ, then by operating with $e_{n-1}^{\max} \cdots e_p^{\max}$ one changes the $\overline{p}$ to $\overline{n}$. This cannot be bracketed below, since there is no unbarred letter below and to the right, so $\varepsilon_n(e_{n-1}^{\max} \cdots e_p^{\max} b) > 0$. Otherwise column ℓ has the form

$$\begin{array}{|c|}\hline \overline{l}_t \\ \hline \vdots \\ \hline \overline{l}_1 \\ \hline n \\ \hline n-1 \\ \hline \vdots \\ \hline k \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|}\hline \overline{l}_t \\ \hline \vdots \\ \hline \overline{l}_1 \\ \hline 0^\alpha \\ \hline n \\ \hline \vdots \\ \hline k \\ \hline \end{array}$$

where $l_i < k$ for all $1 \leq i \leq t$, 0^α stands for α boxes filled with 0, and to the right of this column there are only unbarred letters and possibly $\overline{1}$. By operating with $e_{k-2}^{\max} \cdots e_{l_1}^{\max}$ the rightmost box that is changed is the $\overline{l}_1$ of column ℓ. Applying in addition $e_{n-2}^{\max} \cdots e_{k-1}^{\max}$ column ℓ becomes

$$\begin{array}{|c|}\hline \overline{l}_t \\ \hline \vdots \\ \hline \overline{n-1} \\ \hline n \\ \hline n-2 \\ \hline \vdots \\ \hline k-1 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|}\hline \overline{l}_t \\ \hline \vdots \\ \hline \overline{n-1} \\ \hline 0^\alpha \\ \hline n \\ \hline n-2 \\ \hline \vdots \\ \hline k-1 \\ \hline \end{array}$$

and the columns to the right of column ℓ become

$$\begin{array}{|c|}\hline \overline{1} \\ \hline 0^\beta \\ \hline n \\ \hline n-2 \\ \hline \vdots \\ \hline m \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|}\hline 0^\beta \\ \hline n \\ \hline n-2 \\ \hline \vdots \\ \hline m \\ \hline \end{array}.$$

where $m \geq k-1$ and $\beta \geq 0$. By operating with $e_{n-1}^{\max}$, the letter $\overline{n-1}$ in column ℓ becomes $\overline{n}$ and all the n below and to the right become $n-1$. If there is a 0 it is unbracketed below. If there is no 0, then $\overline{n}$ is unbracketed below. Therefore $\varepsilon_n(e_{n-1}^{\max} \cdots e_{l_1}^{\max} b) > 0$. □

Lemma 5.5. *Let $b \in V^{n,s}$ be J-lowest such that b does not contain any $\overline{k}$ with $k \neq 1$, contains at least one $\overline{1}$ at height n, and has at least one column of height*

strictly less than $n-1$*. Then there exists a sequence* $\vec{a}$ *with elements in* $\{1,\dots,n-1\}$ *such that* $\varepsilon_n(e_{\vec{a}}b)>0$ *and* $\varphi_0(e_{\vec{a}}b)>0$*.*

Proof. Since b is J-lowest, its columns of height n from top to bottom must be $n,n-1,\dots,1$ or $\overline{1},n,n-1,\dots,2$ or for the classical subalgebra B_n possibly $0^\alpha,n,n-1,\dots,1+\alpha$ with $\alpha\ge 1$. Since no more than one 0 can appear in the same row, there can be at most one column of the form $0^\alpha,n,n-1,\dots,1+\alpha$. Furthermore, since b is supposed to contain at least one $\overline{1}$ at height n, we must have $\alpha=1$ if a column containing 0 exists.

Set $k=2\ell+j$ if there is no column of height n containing 0 and $k=2\ell+j-1$ otherwise, where ℓ is the number of columns of height n not of the form $n,n-1,\dots,1$ and j is the number of columns of b of height $n-1$ of the form $n,n-1,\dots,2$. Then $e_1^k(b)$ is a tableau, where the columns of height n not of the form $n,n-1,\dots,1$ consist of the letters $\overline{2},n,n-1,\dots,3,1$ or $0,n,n-1,\dots,3,1$. By passing to the $\{2,\dots,n-1\}$-highest weight element, one obtains a tableau $e_{\vec{a}}b$ without n (except in the columns of height n of the form $n,n-1,\dots,1$), but with $\overline{n}$ or 0 in the last ℓ columns of height n, where $\vec{a}$ has elements in $\{1,\dots,n-1\}$. Hence $\varepsilon_n(e_{\vec{a}}b)>0$. The corresponding $\pm$-diagram possibly has $+$ signs in rows of height n or $n-1$, but no other $+$ signs. Since by assumption there is at least one column of height strictly less than $n-1$, it follows from Proposition 4.10 (2) that $\varphi_0(e_{\vec{a}}b)>0$. □

Lemma 5.6. *Let* $b_1,b_2\in V^{r,s}$ *be* J*-lowest,* $b_1\neq b_2$*,* $\mathrm{wt}\,(b_1)=\mathrm{wt}\,(b_2)$*,* $b_1\sim_{J'}b_2$*, and assume that* b_1 *does not contain any* $\overline{k}\neq\overline{1}$*. If* b_1,b_2 *differ in boxes in rows strictly below* $n-1$*, we have*

$$\begin{aligned}&\varphi_0(e_1^{\max}b_1),\varphi_0(e_1^{\max}b_2)>0 \quad \textit{and}\\ &e_1^{\max}b_1\not\sim_{J'}e_1^{\max}b_2.\end{aligned}$$

Proof. By Corollary 3.3, b_1 and b_2 have the same inner tableau and only differ in the positions of the $\overline{1}$; the number of $\overline{1}$ in b_1 and b_2 must be the same since $\mathrm{wt}\,(b_1)=\mathrm{wt}\,(b_2)$.

By Lemma 3.1, b_1 and b_2 contain $\overline{1}$ in certain positions and the remainder of each column is filled with $n,n-1,\dots,k$ for some k. The $\pm$-diagrams corresponding to b_1 and b_2 contain columns with $-$ (in the position where b_1 and b_2 contain $\overline{1}$, respectively) and otherwise only empty columns. By acting with $e_1^{\max}$ every 2 changes into 1 and every $\overline{1}$ changes to $\overline{2}$. The $\pm$-diagram of $e_1^{\max}b_i$ will possibly have some $+$ at height n and $n-1$, and otherwise only contains empty columns. Therefore the inner shape of $e_1^{\max}b_1$ and $e_1^{\max}b_2$ is different and hence $e_1^{\max}b_1\not\sim_{J'}e_1^{\max}b_2$. By Proposition 4.10 (2) we conclude that $\varphi_0(e_1^{\max}b_1),\varphi_0(e_1^{\max}b_2)>0$. □

Theorem 5.7. *Let* $1\le r\le n$ *for type* $A_{2n}^{(2)}$*, and* $1\le r<n$ *for types* $C_n^{(1)}$ *and* $D_{n+1}^{(2)}$*. We have* $\Psi_0(b)=\Psi_n(b)$ *for all* $b\in V^{r,s}$ *and hence there exists a unique* I*-crystal isomorphism* $\Psi: V^{r,s}\cong B^{r,s}$*.*

Proof. Since $e_i\Psi_0=\Psi_0e_i$ and $e_i\Psi_n=\Psi_ne_i$ for $i\in J$, it suffices to show that $\Psi_n(b)=\Psi_0(b)$ for J-lowest elements b. We prove this by downward induction on $(\mathrm{wt}\,b,\sum_{i=1}^n\epsilon_i)$, where ϵ_i are the canonical basis vectors in $P=\mathbb{Z}^n$ and $(\cdot,\cdot)$ is the canonical inner product; the quantity $(\mathrm{wt}\,b,\sum_{i=1}^n\epsilon_i)$ corresponds to the difference between the number of unbarred and barred letters in the tableau of b. If this value is maximal, there is only one J-lowest element.

First assume that b is a J-lowest element satisfying

$\varphi_0(b) > 0$ and
(1) b contains $\overline{k}$ with $k > 1$ or
(2) b satisfies the conditions of Lemma 5.5.

In case 1 by Lemma 5.4 there exists a sequence $\vec{a}$ consisting of elements in $\{2, \dots, n-1\}$ such that $\varepsilon_n(e_{\vec{a}}b) > 0$. Then we also have $\varphi_0(e_{\vec{a}}b) > 0$. In case 2 by Lemma 5.5 there exists a sequence $\vec{a}$ with elements in $\{1, 2, \dots, n-1\}$ such that $\varphi_0(e_{\vec{a}}b) > 0$ and $\varepsilon_n(e_{\vec{a}}b) > 0$. Set $b' = e_{\vec{a}}b$. We have

$$\begin{aligned} e_n f_0 \Psi_n(b') &= e_n \Psi_n(f_0 b') = e_n \Psi_0(f_0 b') = \Psi_0(e_n f_0 b') \\ &= \Psi_n(f_0 e_n b') = f_0 \Psi_n(e_n b') = f_0 \Psi_0(e_n b') = f_0 e_n \Psi_0(b'). \end{aligned}$$

Here in the 2nd, 4th and 6th equality we have used the induction hypothesis. Hence we have $\Psi_n(b) = \Psi_0(b)$ in this case.

Next assume that the J-lowest element b satisfies

$\varphi_0(b) > 0$ and
b does not contain $\overline{k}$ with $k > 1$ and
b does not satisfy the conditions of Lemma 5.5.

Suppose

$$\Psi_n(b) = \Psi_0(b') \tag{5.2}$$

for b, b' such that $b \neq b'$. One can assume b' is also J-lowest and $\mathrm{wt}\, b = \mathrm{wt}\, b'$. We show by contradiction that this is not possible. We have

$$\Psi_0(b') = e_0 \Psi_n(f_0 b) = e_0 \Psi_0(f_0 b).$$

The second equality is due to the induction hypothesis. From the equality of the LHS and RHS we have $b \sim_{J'} b'$. If b' contains $\overline{k}$ with $k > 1$ or b' satisfies the conditions of Lemma 5.5, we already know $\Psi_n(b') = \Psi_0(b')$ from the previous case. Hence we can assume that b' does not contain $\overline{k}$ with $k > 1$ and does not satisfy the conditions of Lemma 5.5 either.

Suppose that b and b' do not differ in boxes in rows strictly below $n-1$. Since $b \sim_{J'} b'$, this means by Corollary 3.3 that they differ in boxes containing $\overline{1}$ in row n and $n-1$. But then at least one of b or b' satisfies the conditions of Lemma 5.5 which is a contradiction. Hence b and b' must differ in boxes in row strictly below $n-1$. Then by Lemma 5.6 we have $\varphi_0(e_1^{\max} b) > 0$ and

$$e_1^{\max} b \not\sim_{J'} e_1^{\max} b'. \tag{5.3}$$

From (5.2) one has

$$\Psi_0(e_1^{\max} b') = \Psi_n(e_1^{\max} b) = e_0 \Psi_n(f_0 e_1^{\max} b) = e_0 \Psi_0(f_0 e_1^{\max} b).$$

In the last equality we used the induction hypothesis. But the equality of the LHS and RHS contradicts (5.3).

We are left to show $\Psi_n(b) = \Psi_0(b)$ when $\varphi_0(b) = 0$. However, from Lemma 4.6 for type $C_n^{(1)}$ and Lemma 4.9 for types $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ such an element b is unique if we specify the weight. So one has to have $\Psi_n(b) = \Psi_0(b)$ also in this case. This completes the proof. $\square$

6. KR crystals for exceptional nodes

A node r in the Dynkin diagram is called special if there is a Dynkin diagram automorphism that maps r to 0. When r is a special node, the corresponding KR crystal $B^{r,s}$ is irreducible as a $\{1,2,\dots,n\}$-crystal. For type $A_n^{(1)}$, all nodes r are special and have already been treated in Section 4.1. For types $B_n^{(1)}$, $A_{2n-1}^{(2)}$ and $D_n^{(1)}$, the node $r=1$ is special and has already been treated in Section 4.2. The remaining special nodes are $r=n$ for types $C_n^{(1)}$ and $D_{n+1}^{(2)}$, and $r=n-1,n$ for type $D_n^{(1)}$, for which the usual decomposition (1.1) does not hold anymore. The KR crystals for these exceptional nodes are treated in Sections 6.1 and 6.2, respectively.

6.1. $B^{n,s}$ **of type** $C_n^{(1)}, D_{n+1}^{(2)}$**.** In this section we give the combinatorial description of the KR-crystal $B^{n,s}$ of types $C_n^{(1)}$ and $D_{n+1}^{(2)}$. As a $\{1,2,\dots,n\}$-crystal we have the isomorphism

$$B^{n,s} \cong B(s\Lambda_n). \tag{6.1}$$

First consider type $C_n^{(1)}$. The elements in $B(s\Lambda_n)$ are KN-tableaux of shape (s^n) (see Section 2.2). Recall from Section 3.2, that the J'-highest weight elements of shape (s^n) are in bijection with $\pm$-diagrams. Since all columns are of height n and the classical subalgebra of $C_n^{(1)}$ is C_n, each column is either filled with $+$, $-$, or $\mp$. Hence, if there are ℓ_1 columns containing $+$, ℓ_2 columns containing $-$, and ℓ_3 columns containing $\mp$, we may identify $\pm$-diagrams P with triples (ℓ_1,ℓ_2,ℓ_3) such that $\ell_1+\ell_2+\ell_3=s$ and $\ell_1,\ell_2,\ell_3\ge 0$.

In order to describe the affine structure, it suffices to define e_0 on such triples, since e_0 commutes with $e_2,\dots,e_n$. Acting with e_0 changes neither the inner shape of P (since e_0 commutes with $e_2,\dots,e_n$) nor the outer shape of P (by the decomposition (6.1)). Hence ℓ_3 is invariant under e_0. If $\varepsilon_0(\ell_1,\ell_2,\ell_3)>0$, for weight reasons we must have $e_0(\ell_1,\ell_2,\ell_3)=(\ell_1-1,\ell_2+1,\ell_3)$. Again for weight reasons, if $\ell_1=0$, then $e_0(\ell_1,\ell_2,\ell_3)=\emptyset$. We now calculate $\varepsilon_0(\ell_1,\ell_2,\ell_3)$.

Lemma 6.1. $\varepsilon_0(\ell_1,\ell_2,\ell_3)=\ell_1$.

Proof. If $\ell_2=0$, then $\varepsilon_0(\ell_1,0,\ell_3)-\varphi_0(\ell_1,0,\ell_3)=\ell_1$ and therefore $\varepsilon_0(\ell_1,0,\ell_3)\ge\ell_1$. But by the previous observation $\varepsilon_0(\ell_1,\ell_2,\ell_3)\le\ell_1$. The claim follows. □

Definition 6.1. The combinatorial crystal $V^{n,s}$ of type $C_n^{(1)}$ is defined to be $B(s\Lambda_n)$ as a $\{1,2,\dots,n\}$-crystal. The action of f_0 and e_0 on $\{2,3,\dots,n\}$-highest weight elements is given by

$$f_0(\ell_1,\ell_2,\ell_3)=\begin{cases}(\ell_1+1,\ell_2-1,\ell_3) & \text{if } \ell_2>0,\\ \emptyset & \text{otherwise,}\end{cases}$$

$$e_0(\ell_1,\ell_2,\ell_3)=\begin{cases}(\ell_1-1,\ell_2+1,\ell_3) & \text{if } \ell_1>0,\\ \emptyset & \text{otherwise.}\end{cases}$$

Next consider type $D_{n+1}^{(2)}$ whose classical subalgebra is of type B_n. Since $\Lambda_n=\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n)$, the elements in $B(s\Lambda_n)$ are KN-tableaux of shape $((s/2)^n)$ when s is even and of shape $(((s-1)/2)^n)$ plus an extra spin column when s is odd. By Section 3.2, the J'-highest weight elements are in bijection with $\pm$-diagrams, where

columns of height n can contain $+$, $-$, $\mp$ and at most one 0; the spin column of half width can contain $+$ or $-$.

We may again encode a $\pm$-diagram P as a triple (ℓ_1, ℓ_2, ℓ_3), where ℓ_1 is twice the number of columns containing a single $+$ sign, ℓ_2 is twice the number of columns containing a single $-$ sign (where spin column are counted as $1/2$ columns), and ℓ_3 is twice the number of columns containing $\mp$. If P contains a 0-column, then $\ell_1 + \ell_2 + \ell_3 = s - 2$, otherwise $\ell_1 + \ell_2 + \ell_3 = s$.

As in the case $C_n^{(1)}$, since e_0 commutes with $e_2, \dots, e_n$ it suffices to specify the action of e_0 on $\{2, 3, \dots, n\}$-components or equivalently on triples (ℓ_1, ℓ_2, ℓ_3).

Lemma 6.2. *$\varepsilon_0(\ell_1, \ell_2, \ell_2) = \ell_1 + \gamma$, where γ is 1 if there is a 0 column, and 0 otherwise.*

Proof. The proof is the same as above for $C_n^{(1)}$. □

Definition 6.2. The combinatorial crystal $V^{n,s}$ of type $D_{n+1}^{(2)}$ is defined to be $B(s\Lambda_n)$ as a $\{1, 2, \dots, n\}$-crystal. The action of f_0 and e_0 on $\{2, 3, \dots, n\}$-highest weight elements is given by

$$f_0(\ell_1, \ell_2, \ell_3) = \begin{cases} (\ell_1 + 2, \ell_2, \ell_3) & \text{if } \ell_1 + \ell_2 + \ell_3 < s, \\ (\ell_1, \ell_2 - 2, \ell_3) & \text{if } \ell_1 + \ell_2 + \ell_3 = s \text{ and } \ell_2 > 1, \\ (\ell_1 + 1, 0, \ell_3) & \text{if } \ell_1 + \ell_2 + \ell_3 = s \text{ and } \ell_2 = 1, \\ \emptyset & \text{if } \ell_1 + \ell_2 + \ell_3 = s \text{ and } \ell_2 = 0, \end{cases}$$

$$e_0(\ell_1, \ell_2, \ell_3) = \begin{cases} (\ell_1, \ell_2 + 2, \ell_3) & \text{if } \ell_1 + \ell_2 + \ell_3 < s, \\ (\ell_1 - 2, \ell_2, \ell_3) & \text{if } \ell_1 + \ell_2 + \ell_3 = s \text{ and } \ell_1 > 1, \\ (0, \ell_2 + 1, \ell_3) & \text{if } \ell_1 + \ell_2 + \ell_3 = s \text{ and } \ell_1 = 1, \\ \emptyset & \text{if } \ell_1 + \ell_2 + \ell_3 = s \text{ and } \ell_1 = 0. \end{cases}$$

Theorem 6.3. *For types $C_n^{(1)}$ and $D_{n+1}^{(2)}$, we have*

$$V^{n,s} \cong B^{n,s}.$$

Proof. By [28, Theorem 1.1] the KR crystal $B^{n,s}$ exists. Given the decomposition (6.1) as a $\{1, 2, \dots, n\}$-crystal, the construction of $V^{n,s}$ as an affine crystal was uniquely specified by weight considerations. Hence, since $V^{n,s}$ and $B^{n,s}$ have the same $\{1, 2, \dots, n\}$-decomposition and $B^{n,s}$ exists, $V^{n,s}$ and $B^{n,s}$ must be isomorphic. □

6.2. **$B^{n,s}$ and $B^{n-1,s}$ of type $D_n^{(1)}$.** In this section we give a combinatorial model for the KR-crystals $B^{n,s}$ and $B^{n-1,s}$ of type $D_n^{(1)}$ that are associated to the spin nodes $n-1$ and n in the Dynkin diagram. As $\{1, 2, \dots, n\}$-crystals we have the isomorphisms

$$\begin{aligned} B^{n,s} &\cong B(s\Lambda_n), \\ B^{n-1,s} &\cong B(s\Lambda_{n-1}). \end{aligned} \tag{6.2}$$

The combinatorial KR-crystals $V^{n,s}$ and $V^{n-1,s}$ are constructed to have the same classical decomposition as in (6.2). To define the affine crystal action, we first introduce an involution $\sigma : B^{n,s} \leftrightarrow B^{n-1,s}$ corresponding to the Dynkin diagram automorphism that interchanges the nodes 0 and 1. Under this involution, $\{2, 3, \dots, n\}$-components need to be mapped to $\{2, 3, \dots, n\}$-components. Hence it suffices to

define σ on $\{2,3,\dots,n\}$-highest weight elements or equivalently $\pm$-diagrams. Recall from Section 3.2, that for weights $\Lambda = s\Lambda_n$ or $s\Lambda_{n-1}$, the $\pm$-diagram can contain columns with $+$ and $\mp$ or with $-$ and $\mp$ (but not a mix of $-$ and $+$ columns).

Definition 6.3. The involution $\sigma : B^{n,s} \leftrightarrow B^{n-1,s}$ maps a $\pm$-diagram P to a $\pm$-diagram P' of opposite color where columns containing $+$ are interchanged with columns containing $-$ and vice versa.

Definition 6.4. The combinatorial crystal $V^{n,s}$ (resp. $V^{n-1,s}$) of type $D_n^{(1)}$ is defined to be $B(s\Lambda_n)$ (resp. $B(s\Lambda_{n-1})$) as a $\{1,2,\dots,n\}$-crystal. The action of f_0 and e_0 is

$$e_0 = \sigma \circ e_1 \circ \sigma \qquad \text{and} \qquad f_0 = \sigma \circ f_1 \circ \sigma$$

with σ as in Definition 6.3.

Theorem 6.4. *For type $D_n^{(1)}$, we have*

$$V^{n,s} \cong B^{n,s},$$
$$V^{n-1,s} \cong B^{n-1,s}.$$

Proof. $V^{n,s}$ and $B^{n,s}$ have the same decomposition as $\{1,2,\dots,n\}$ and $\{0,2,\dots,n\}$-crystals

$$\begin{aligned} \Psi_0 : \quad & V^{n,s} \cong B^{n,s} \cong B(s\Lambda_n) && \text{as a } \{1,2,\dots,n\}\text{-crystals,} \\ \Psi_1 : \quad & V^{n,s} \cong B^{n,s} \cong B(s\Lambda_{n-1}) && \text{as a } \{0,2,\dots,n\}\text{-crystals.} \end{aligned}$$

The $\{1,2,\dots,n\}$-crystal isomorphism $V^{n,s} \cong B(s\Lambda_n)$ is true by definition and the $\{0,2,\dots,n\}$-crystal isomorphism $V^{n,s} \cong B(s\Lambda_{n-1})$ follows by the application of σ since $e_0 = \sigma e_1 \sigma$ and $e_i = \sigma e_i \sigma$ for $i \neq 0,1$. Chari [2] proved that $B^{n,s} \cong B(s\Lambda_n)$ as $\{1,2,\dots,n\}$-crystals. For the proof that $B^{n,s} \cong B(s\Lambda_{n-1})$ as a $\{0,2,\dots,n\}$-crystal, it suffices to show there exists a corresponding highest weight vector, since the crystal is irreducible. Applying the Weyl group element r_β, which is the reflection for the root $\beta = \epsilon_1 + \epsilon_n$, to the $\{1,2,\dots,n\}$-highest weight element yields the $\{0,2,\dots,n\}$-highest weight element.

Note that if $\Psi_0(b) = \Psi_1(b)$ for a b in a given D_{n-1}-component $\mathcal{C}$, then $\Psi_0(b') = \Psi_1(b')$ for all $b' \in \mathcal{C}$ since $e_i\Psi_0(b') = \Psi_0(e_ib')$ and $e_i\Psi_1(b') = \Psi_1(e_ib')$ for $i \in J' = \{2,3,\dots,n\}$. Furthermore observe that by Remark 5.1 Ψ_0 and Ψ_1 preserve weights, that is, $\mathrm{wt}\,(b) = \mathrm{wt}\,(\Psi_0(b)) = \mathrm{wt}\,(\Psi_1(b))$ for all $b \in V^{n,s}$.

Since e_i commutes with Ψ_0 and Ψ_1 for $i \in J'$, it follows that J'-components in $V^{n,s}$ must map to J'-components in $B^{n,s}$. However, as can be seen from the description using $\pm$-diagrams, the branching $D_n \to D_{n-1}$ on $B(s\Lambda_n)$ and $B(s\Lambda_{n-1})$ is multiplicity-free once the weight is fixed. Hence we must have $\Psi_0(b) = \Psi_1(b)$ for all $b \in V^{n,s}$. The proof for $V^{n-1,s}$ is analogous. □

7. Dynkin automorphism for type $C_n^{(1)}$ and $D_{n+1}^{(2)}$

By construction, the Dynkin diagram automorphism for type $A_{n-1}^{(1)}$, $B_n^{(1)}$, $D_n^{(1)}$, and $A_{2n-1}^{(2)}$ acts on the combinatorial crystal $V^{r,s}$, except for $r = n-1, n$ for type $D_n^{(1)}$. The Dynkin diagrams for type $C_n^{(1)}$ and $D_{n+1}^{(2)}$ also have an automorphism mapping $i \mapsto n-i$ for all $i \in \{0,1,\dots,n\}$. However, from the construction of $V^{r,s}$ for these types using Dynkin diagram foldings and similarity methods, it is not obvious that this Dynkin diagram automorphism extends to $V^{r,s}$. This is proven in Theorem 7.1. This shows in particular that [7, Assumption 1] holds, which was

used to show that the classical isomorphism from the Demazure crystal to the KR crystal, sends zero arrows to zero arrows.

Theorem 7.1. *Let $B^{r,s}$ $(1 \le r \le n, s \ge 1)$ be the KR crystal for type $C_n^{(1)}, D_{n+1}^{(2)}$. Then there exists an involution σ on $B^{r,s}$ satisfying*

$$\sigma \circ e_i = e_{n-i} \circ \sigma \quad \text{and} \quad \sigma \circ f_i = f_{n-i} \circ \sigma \quad \text{for all } i \in I. \tag{7.1}$$

Let $\widetilde{\mathcal{B}}^{r,s}$ be the fixed point crystal in Case (a) of Section 2.1 constructed in [25]. Naito and Sagaki [25] show that $\widetilde{\mathcal{B}}^{r,s}$ is a regular $D_{n+1}^{(2)}$-crystal which decomposes into $\bigoplus_\Lambda B(\Lambda)$ for $1 \le r < n$ and $B(s\Lambda_n)$ for $r = n$ as a B_n-crystal, where the direct sum $\bigoplus_\Lambda$ is over all Λ obtained from $s\Lambda_r$ by removing boxes and $B(\Lambda)$ is a highest weight B_n-crystal of highest weight Λ. They also show that as a $D_{n+1}^{(2)}$-crystal it is isomorphic to the virtual $U_q'(D_{n+1}^{(2)})$-crystal defined in [29, Section 6.7].

The next lemma is used for the proof of type $D_{n+1}^{(2)}$.

Lemma 7.2.

(1) *The decomposition*

$$\widetilde{\mathcal{B}}^{r,s} \simeq \begin{cases} \bigoplus_\Lambda B(\Lambda) & \text{for } 1 \le r < n \\ B(s\Lambda_n) & \text{for } r = n \end{cases}$$

also holds as a $\{0, 1, \dots, n-1\}$-crystal.

(2) *There exists an involution σ on $\widetilde{\mathcal{B}}^{r,s}$ satisfying* (7.1).

Proof. Note that the Weyl group of type B_n contains an element that maps $\Lambda_j - 2\Lambda_0$ to $\Lambda_{n-j} - 2\Lambda_n$ for all $1 \le j \le n$. Hence, the decomposition as a $\{0, 1, \dots, n-1\}$-crystal follows from that as a $\{1, 2, \dots, n\}$-crystal as in the proof of Lemma 4.6.

We prove (2). Set $\sigma = \mathrm{pr}^n$, where pr is the promotion operator on the ambient $A_{2n-1}^{(1)}$-crystal as defined in Section 4.1. The map σ satisfies (7.1) with $x_i = \hat{x}_i$ for $i = 0, n$ and $x_i = \hat{x}_i \hat{x}_{2n-i}$ otherwise, where $x = e, f$. Hence, it suffices to show

$$\sigma(\widetilde{\mathcal{B}}^{r,s}) \subset \widetilde{\mathcal{B}}^{r,s}. \tag{7.2}$$

Let u be the unique dominant extremal element in the ambient $A_{2n-1}^{(1)}$-crystal. Then u belongs to $\widetilde{\mathcal{B}}^{r,s}$. The inclusion (7.2) is now clear, since $\widetilde{\mathcal{B}}^{r,s}$ is generated from u by applying e_i and f_i. □

Proof of Theorem 7.1 for type $D_{n+1}^{(2)}$. By Theorem 5.7 we know $B^{r,s} \simeq \widetilde{\mathcal{B}}^{r,s}$ as $D_{n+1}^{(2)}$-crystals. Hence σ exists by (2) of Lemma 7.2. □

We prepare a lemma and a proposition for the proof of type $C_n^{(1)}$.

Lemma 7.3. *Let $V^{r,s}_{C_n^{(1)}}$ be the $C_n^{(1)}$-crystal constructed in Section 4.3. Let b be a $\{2, \dots, n\}$-highest element of $V^{r,2s}_{C_n^{(1)}}$ whose shape is obtained from an $r \times 2s$ rectangle by removing 1×4 rectangular pieces. Let P be the corresponding $\pm$-diagram of b. Suppose that the number of columns with $\mp, +, -, \cdot$ at each height is even. Then the shape of $e_0^2(b)$ and the corresponding $\pm$-diagram have the same property.*

Proof. Let $\hat{V}^{r,s}$ be the ambient crystal of type $A_{2n-1}^{(2)}$ in the definition of the type $C_n^{(1)}$ KR crystal. We follow the same set-up as in the proof of Lemma 4.7. For

elements in $b \in V^{r,s} := V^{r,s}_{C_n^{(1)}}$, we have $\sigma(b) = b$. Hence on $V^{r,s}$ we have $e_0^2 = \hat{e}_1\hat{e}_0\hat{e}_1\hat{e}_0 = \hat{e}_1^2\hat{e}_0^2 = \hat{e}_1^2(\sigma\hat{e}_1\sigma)(\sigma\hat{e}_1\sigma) = \hat{e}_1^2\sigma\hat{e}_1^2$. The proof follows that of Lemma 4.7 with all columns and operations doubled. $\square$

Proposition 7.4.

(1) *There exists a $C_n^{(1)}$-crystal $\widetilde{V}^{r,s}$ and a unique injective map $S : \widetilde{V}^{r,s} \longrightarrow V^{r,s}_{D^{(2)}_{n+1}}$ such that*

$$S(e_i b) = e_i^{m_i} S(b), \quad S(f_i b) = f_i^{m_i} S(b) \text{ for } i \in I,$$

where $(m_0, m_1, \dots, m_{n-1}, m_n) = (2, 1, \dots, 1, 2)$.

(2) $\widetilde{V}^{r,s}$ *is connected.*

(3) *There exists an involution σ on $\widetilde{V}^{r,s}$ satisfying* (7.1).

Proof. We only prove the case $1 \le r \le n-1$. The case $r = n$ is similar and easier.

Let us prove (1). Let $B(\Lambda)$ (resp. $B_{B_n}(\Lambda)$) be the C_n (resp. B_n)-crystal of the highest weight module of highest weight Λ. By [20, Theorem 5.1], there exists a unique injective map $\overline{S} : B(\Lambda) \to B_{B_n}(\Lambda)$ such that $\overline{S}(e_i b) = e_i^{m_i}\overline{S}(b)$ and $\overline{S}(f_i b) = f_i^{m_i}\overline{S}(b)$ for $i \in I_0 = \{1, 2, \dots, n\}$.

Now define $\widetilde{V}^{r,s}$, as a C_n-crystal, by

$$\widetilde{V}^{r,s} = \bigoplus_{\Lambda} B(\Lambda), \tag{7.3}$$

where the sum is over all Λ obtained from $s\Lambda_r$ by removing horizontal dominoes. From the above explanation we have an injective map $S : \widetilde{V}^{r,s} \to V^{r,s}_{D^{(2)}_{n+1}}$. We introduce the 0-action on $\widetilde{V}^{r,s}$. Recall the construction of an injective map S (denoted here by S') : $V^{r,s}_{D^{(2)}_{n+1}} \to V^{r,2s}_{C_n^{(1)}}$ of Section 4.4. For a $\{2, \dots, n\}$-highest element b in $V^{r,s}_{D^{(2)}_{n+1}}$, it is shown that $S'S(b)$ satisfies the assumptions of Lemma 7.3 by using Lemma 3.5. Hence, $e_0^2 S'S(b)$ also has the same property, so there exists a $b' \in \widetilde{V}^{r,s}$ such that $S'S(b') = e_0^2 S'S(b)$. One can define $e_0(b) = b'$ on $\widetilde{V}^{r,s}$ and we have $S(e_0 b) = e_0^2 S(b)$. The case f_0 is similar.

Next we prove (2). Suppose that $\widetilde{V}^{r,s}$ is not connected. Then there must exist Λ, Λ' in the decomposition (7.3) that differ only in one horizontal domino, but lie in different components (otherwise all Λ would lie in the same component and hence $\widetilde{V}^{r,s}$ would be connected). By Lemma 4.7 there exists a $\{2, 3, \dots, n\}$-highest weight element $b \in V^{r,s}$ of type $C_n^{(1)}$ such that $b \in B(\Lambda)$ and $b' := e_0(b) \in B(\Lambda')$. Identifying the classical decomposition of $V^{r,s}$ as in Lemma 4.5 and $\widetilde{V}^{r,s}$ as in (7.3), we may consider the corresponding elements $\tilde{b} \in B(\Lambda) \subset \widetilde{V}^{r,s}$ and $\tilde{b}' \in B(\Lambda') \subset \widetilde{V}^{r,s}$. Then $S'S(\tilde{b}) \in B(2\Lambda)$ and $S'S(\tilde{b}') \in B(2\Lambda')$, and using Lemma 3.5 for the corresponding $\pm$-diagrams we find that $S'S(\tilde{b}') = e_0^2 S'S(\tilde{b})$. This implies by the definition of e_0 on $\widetilde{V}^{r,s}$ that $\tilde{b}' = e_0(\tilde{b})$ which contradicts the assumption that $B(\Lambda)$ and $B(\Lambda')$ lie in different components of $\widetilde{V}^{r,s}$. Hence $\widetilde{V}^{r,s}$ must be connected.

Finally we prove (3). By (1) one can consider the problem in the image of S. By Theorem 7.1 for type $D^{(2)}_{n+1}$ we know that there exists an involution σ satisfying (7.1) on $V^{r,s}_{D^{(2)}_{n+1}}$. Since Im S is generated by $e_i^{m_i}$ using (2), it is clear that Im S is closed under σ. $\square$

Proof of Theorem 7.1 for type $C_n^{(1)}$. By construction, $\widetilde{V}^{r,s}$ of Proposition 7.4 and $B^{r,s}$ have the same decomposition as $\{1,\dots,n\}$-crystals. From Proposition 7.4 (3) it follows that $\widetilde{V}^{r,s}$ and $B^{r,s}$ have the same decomposition also as $\{0,\dots,n-1\}$-crystals. Then by Theorem 5.7 we know that $B^{r,s} \simeq \widetilde{V}^{r,s}$ as $C_n^{(1)}$-crystals. Hence σ on $B^{r,s}$ exists by (3) of Proposition 7.4. □

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